Deligne-Serre lifting lemma

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We'll prove the following lemma.

Theorem 1. ***(Deligne-Serre lifting Lemma)*** *Let $\mathfrak{O}$ be a discrete valuation ring of fraction field K, maximal ideal m and residue field $k = \mathfrak{O}/m$. Let M be a finite free $\mathfrak{O}$-module and $\mathcal{T}$ be a set of commuting family of $\mathfrak{O}$-endomorphism of M. Let $f \in M/mM = M \otimes k$ a nonzero common eigenvector for $T \in \mathcal{T}$ with eigenvalue $a_T \in k$. Then there exists a discrete valuation ring $\mathfrak{O}'$ containing and finite over $\mathfrak{O}$, with maximal ideal m' such that $\mathfrak{O} \cap m' = m$ and nonzero element f' in*

$$M' = \mathfrak{O}' \otimes_{\mathfrak{O}} M$$

which is a common eigenvector with eigenvalue $a_{T'}$ such that $a_{T'} \equiv a_T (\text{mod } m')$.

Let $\mathcal{H}$ be a subalgebra of $End(M)$ generated by $\mathcal{T}$. Since M is a free $\mathfrak{O}$-module of rank r, $End(M)$ is free $\mathfrak{O}$-module of rank r^2. So as a subalgebra of $End(M)$, $\mathcal{H}$ is also free $\mathfrak{O}$-module of finite rank. Choosing a basis for $\mathcal{H}$, we may assume $\mathcal{T}$ is a finite set $\mathcal{T} = \{T_1, T_2, \cdots T_n\}$. Let m_{T_i} be a minimal polynomial for T_i. We can find a finite field extension K' of K so that every m_{T_i} splits. By replacing K,$\mathfrak{O}$ by K',$\mathfrak{O}'$, we may assume $K \otimes \mathcal{H}$ is an artinian ring with every residue field is K.
Let $\chi : \mathcal{H} \to k$ be a homomorphism such that $h \cdot f = \chi(h) f$ for all $h \in \mathcal{H}$. [1] Since $\mathcal{H}$ is free over $\mathfrak{O}$, there exists a prime ideal $\mathfrak{p} \subseteq \mathcal{H}$ contained in a

[1] $T \cdot f = \overline{T(\widetilde{f})} = a_T f$, $a \cdot f = \overline{a} f$ where $a \in \mathfrak{O}$ and $\overline{a} = a \bmod m \in \mathfrak{O}/m$. This is well-defined since $T(mM) = mT(M) \subset mM$

maximal ideal $ker(\chi)$ and $\mathfrak{p} \cap \mathfrak{D} = 0$ [2] Note that $\mathcal{H}/\mathfrak{p}$ is a finitely generated $\mathfrak{D}$-module and also a domain containing $\mathfrak{D}$. Using this $\mathfrak{p}$, define

$$\chi' : \mathcal{H} \to \mathfrak{D}$$

Then we have $\chi \equiv \chi' \bmod m$. [3] The prime ideal $\mathfrak{P}$ of $K \otimes \mathcal{H}$ generated by $\mathfrak{p}$ belongs to the support of $K \otimes M$, which is equivalent that it contains the annihilator of $K \otimes M$. Here're some facts from the commutative algebra

Definition 0.0.1. *Let R be a ring and let M be an R-module. A prime P of R is associated to M if P is the annihilator of an element of M. The set of all primes associated to M is written $Ass_R(M)$ or simply $Ass(M)$.*

Definition 0.0.2. *The support of M, written $Supp(M)$ is the set of prime ideals such that $M_P \neq 0$ If M is a finitely generated R-module and P is a prime of R, $P \in Supp(M)$ iff P contains the annihilator of M.*

Fact 1. *Let R be a Noetherian ring and M be a finite R-module. Then the set of minimal elements of $Ass(M)$ and $Supp(M)$ coincide*

By this fact, we have $\mathfrak{P} \in Ass(K \otimes M)$. So we have a nonzero element $f'' \in K \otimes M$, which is annihilated by $\mathfrak{P}$, that is, $hf'' = \chi'(h)f''$. Then one can take f' as nonzero multiple of f''.

Comments We have commuting endomorphisms $T_1, .., T_n$. Extending K, we may assume every eigenvalues are in K. Find a basis $(e_1, e_2, ..., e_r)$ of M so that matrix for each T_i is upper triangular.

References 1. Deligne, P., Serre, J-P. : Formes modulaires de poids 1. Ann. Scient. Ec. Norm, Sup.,4^e série, 7, 507-530(1974)
2. David Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry

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[2] If R is a commutative ring with 1, then R has minimal prime ideal and such prime ideal is a subset of the zero-divisors of R. Since $\mathcal{H}$ is free over $\mathfrak{D}$, element of $\mathfrak{D}$ is not a zero divisor

[3] This follows from $\chi'(ker\chi) \subseteq m$ which is a consequence of the lying over theorem.