

# PERIOD MATRICES OF HYPERELLIPTIC CURVES

Bernhard Schindler

## 1 Introduction

To fix notation let me first recall the definition of a period matrix of a hyperelliptic curve.

Let  $C$  be a hyperelliptic curve over  $\mathbf{C}$  of genus  $g > 1$ . Then  $H^0(C, \Omega_C) \cong \mathbf{C}^g$  and  $H_1(C, \mathbf{Z}) \cong \pi_1(C, p_0)^{\text{ab}} \cong \mathbf{Z}^{2g}$  and we get an embedding

$$\begin{array}{ccc} H_1(C, \mathbf{Z}) & \longrightarrow & H^0(C, \Omega_C)^* \\ \gamma & \longmapsto & \int_\gamma \end{array}$$

such that the image is a lattice in  $H^0(C, \Omega_C)^*$ . The quotient

$$\text{Jac}(C) = H^0(C, \Omega_C)^* / H_1(C, \mathbf{Z})$$

is a complex torus, allowing a principal polarisation defined by a hermitian form on  $H^0(C, \Omega_C)$

$$(\omega_1, \omega_2) \longmapsto \int \omega_1 \wedge \bar{\omega}_2$$

or equivalently by a symplectic form on  $H_1(C, \mathbf{Z})$

$$(\gamma_1, \gamma_2) \longmapsto \gamma_1 \cdot \gamma_2,$$

the intersection product. The importance of  $\text{Jac}(C)$  is underlined by the following theorem.

**Theorem 1 (Torelli)** *The principally polarized abelian variety  $\text{Jac}(C)$  determines the curve  $C$  upto isomorphism.*

If a symplectic basis  $\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g$  of  $H_1(C, \mathbf{Z})$  and a corresponding basis  $\omega_1, \dots, \omega_g$  of  $H^0(C, \Omega_C)$  such that

$$\int_{\gamma_j} \omega_i = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

is chosen, the principal polarized abelian variety  $\text{Jac}(C)$  is determined by the *period matrix*  $\Pi = (\pi_{i,j}) = (\int_{\delta_j} \omega_i)$ . Riemann's relations say that  $\Pi \in \mathfrak{H}_g = \{Z \in M_g(\mathbf{C}) : Z^t = Z \text{ and } \text{Im}Z > 0\}$ . In 1888 Bolza [1] succeeded in describing period matrices for hyperelliptic curves of genus 2 with extra automorphisms. In this paper it will be shown how a period matrix can be determined for a given hyperelliptic curve  $C$  of arbitrary genus possessing an automorphism of reduced order of at least  $2g$ .

For this, let  $\sigma$  be an automorphism of  $C$  and consider its representations  $\sigma^*$  on  $H^0(C, \Omega_C)$  and  $\sigma_*$  on  $H_1(C, \mathbf{Z})$ . With respect to the bases considered above  $\sigma^*$  and  $\sigma_*$  can be described by matrices  $L \in \text{Gl}_g(\mathbf{C})$  and

$$M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \text{Sp}_g(\mathbf{Z}).$$

The transformation formula  $\int_{\sigma_*\gamma} \omega = \int_{\gamma} \sigma^*\omega$  will then result in equations for  $\Pi$ :

$$\Pi = (A\Pi + B)(C\Pi + D)^{-1} \tag{AEX}$$

and

$$L = C\Pi + D \tag{DII}$$

The hyperelliptic curve given by the equation  $w^2 = p(z)$  where  $(\deg p = 2g + 2 \text{ or } 2g + 1)$  has a simple basis of  $H^0(C, \Omega_C)$ :

$$\frac{dz}{w}, z \frac{dz}{w}, \dots, z^{g-1} \frac{dz}{w}.$$

If  $L' \in \text{Gl}_g(\mathbf{C})$  describes  $\sigma^*$  with respect to this basis, we get

$$\sigma_i(L') = \sigma_i(C\Pi + D) \quad (i = 1, \dots, g) \tag{DIS}$$

where  $\sigma_i(L)$  are the coefficients of the characteristic polynomial of  $L$ .

All automorphisms of a hyperelliptic curve  $C \xrightarrow{2:1} \mathbf{P}^1$  are liftings of automorphisms of  $\mathbf{P}^1$  operating on the set of  $2g + 2$  branch points. Let  $G = \text{Aut}(C) = \text{Aut}(C)/\langle \iota \rangle < \iota > \subset \text{PSL}_2(\mathbf{C})$ ,  $\iota$  the hyperelliptic involution, be the reduced automorphism group. The operation of the automorphism group  $\text{Aut}(C)$  on  $C$  can be described by the number of  $G$ -orbits of branch points in  $\mathbf{P}^1$  of a given length. We get the following types of hyperelliptic curves with automorphism structure having  $s$  moduli (see [2]).

type	$G$	$g$	restrictions
$Z_r(s, p) \ (r \geq 2)$	$Z_r$	$\frac{(s+1)r+p}{2} - 1$	$(s+1)r + p \equiv 0 \pmod{2}$ $p \in \{0, 1, 2\}$
$D_r(s, q, p) \ (r \geq 3)$	$D_r$	$sr + p + \frac{qr}{2} - 1$	$qr \equiv 0 \pmod{2}$ $q \in \{0, 1, 2\}, p \in \{0, 1\}$
$V_4(s, q)$	$V_4$	$2s + q - 1$	$q \in \{0, 1, 2, 3\}$
$A_4(s, k, e)$	$A_4$	$6s + 3k + 2e - 1$	$k \in \{0, 1\}, e \in \{0, 1, 2\}$
$S_4(s, k, f, e)$	$S_4$	$12s + 6k + 3f + 4e - 1$	$k, e, f \in \{0, 1\}$
$A_5(s, k, f, e)$	$A_5$	$30s + 15k + 10f + 6e - 1$	$k, e, f \in \{0, 1\}$

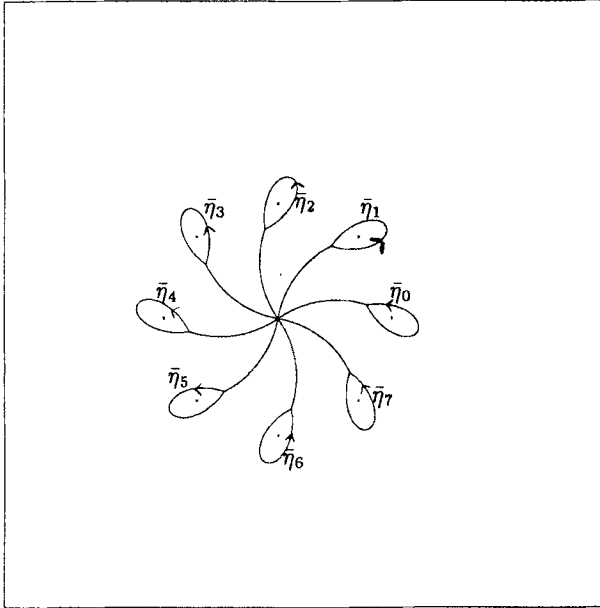


Figure 1: Curves of type  $D_8(0,1,0)$  ( $g = 3$ )

Thus eventually ( $g > 30$ ) only for the cases  $D_{2g+2}(0,1,0)$ ,  $Z_{2g+1}(0,1)$  and  $D_{2g}(0,1,1)$  there will be just a single curve of this type. So these seem to be the most rewarding cases to be considered.

Let  $\zeta_n = \exp(\frac{2\pi i}{n})$  denote the  $n$ -th root of unity. For

$$M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \text{Sp}_g(\mathbf{Z})$$

we set

$$M^n = \begin{pmatrix} D_n & C_n \\ B_n & A_n \end{pmatrix}.$$

I wish to express my cordial thanks to my teacher Prof. H. Lange for his advice and encouragement.

## 2 Curves of type $D_{2g+2}(0,1,0)$

**Theorem 2** *The period matrix of the hyperelliptic curve with equation  $w^2 = z^{2g+2} - 1$  can be given by*

$$\pi_{k,j} = \frac{1}{g+1} \left( \sum_{\nu=1}^k \frac{1 + \cos \frac{2\nu-1}{g+1} \pi}{\sin \frac{2\nu-1}{g+1} \pi} + \frac{1 + \cos \frac{2(j-\nu)+1}{g+1} \pi}{\sin \frac{2(j-\nu)+1}{g+1} \pi} \right) i \quad (k \leq j).$$

Proof. On the curve  $C$  we consider paths  $\eta_j$  that are liftings of the paths  $\bar{\eta}_j$  in figure 1, such that

$$\eta_j^2 \sim 1 \quad j = 0, \dots, 2g+1$$

and

$$\eta_0 \cdots \eta_{2g+1} \sim 1.$$

A symplectic basis can be given by ( $i = 1, \dots, g$ )

$$\begin{aligned} \gamma_i &= [\eta_{2i-1}\eta_{2i}], \\ \delta_i &= [\eta_{2i-1} \cdots \eta_0]. \end{aligned}$$

The following two automorphisms operate on  $C$ :

$$T(z, w) = (\zeta_{2g+2}z, \pm w)$$

and

$$T_1(z, w) = \left(\frac{1}{z}, \pm z^{-(g+1)}iw\right).$$

The operation of  $T$  on  $H_1(C, \mathbf{Z})$  is given by

$$T_* : \begin{cases} \eta_j & \mapsto \eta_{j+1} \quad (j = 0, \dots, 2g) \\ \eta_{2g+1} & \mapsto \eta_0 \end{cases}$$

thus

$$\begin{aligned} T_*\gamma_i &= [\eta_{2i}\eta_{2i+1}] = \begin{cases} \delta_i - \delta_{i+1} & i < g \\ \delta_g & i = g \end{cases} \\ T_*\delta_i &= [\eta_{2i}\eta_{2i-1} \cdots \eta_1] = -\gamma_1 - \gamma_2 - \cdots - \gamma_i \end{aligned}$$

or

$$M = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 \\ -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 & \vdots & & & \ddots & \vdots \\ -1 & \cdots & \cdots & \cdots & -1 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \in \text{Sp}_g(\mathbf{Z}).$$

The operation on  $H^0(C, \Omega_C)$  is described by

$$L' = \begin{pmatrix} \zeta_{2g+2} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \zeta_{2g+2}^2 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \zeta_{2g+2}^{g-1} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \zeta_{2g+2}^g \end{pmatrix}.$$

Then

$$M^2 = \begin{pmatrix} D_2 & 0 \\ 0 & A_2 \end{pmatrix}$$

and  $M^2(\Pi) = A_2 \Pi D_2^{-1} = \Pi$  yields the following equations

$$\begin{aligned} \pi_{i,j} &= \pi_{1,1} - \pi_{i+1,1} - \pi_{1,j+1} + \pi_{i+1,j+1} \quad (i, j < g) \\ \pi_{g,j} &= \pi_{1,1} - \pi_{1,j+1} \quad (j < g) \\ \pi_{i,g} &= \pi_{1,1} - \pi_{i+1,1} \quad (i < g) \\ \pi_{g,g} &= \pi_{1,1}. \end{aligned}$$

Substituting  $\alpha_i := \pi_{1,i}$  one gets

$$\pi_{k,i} = -k\alpha_1 + \sum_{j=1}^k \alpha_j + \sum_{j=1}^k \alpha_{i-j+1} \quad (k \leq i).$$

And the  $\alpha_i$ 's satisfy the following relations

$$\begin{aligned} \alpha_{i+1} + \alpha_{g-i+1} &= \alpha_1 \quad (i = 1, \dots, g-1) \\ 2 \sum_{i=1}^g \alpha_i &= (g+1)\alpha_1. \end{aligned}$$

On the other hand

$$\begin{aligned} \text{Tr}(C\Pi + D) &= \frac{g+1}{2}\alpha_1 \\ \text{Tr}(C_{2i+1}\Pi + D_{2i+1}) &= \frac{g+1}{2}(-\alpha_1 + 2\alpha_{i+1}) \quad (i = 1, \dots, g-1) \end{aligned}$$

and for  $i = 1, \dots, g-1$

$$\text{Tr}(L^{2i+1}) = \frac{1 + \zeta_{2g+2}^{2i+1}}{1 - \zeta_{2g+2}^{2i+1}}$$

So we get for  $\alpha_i$  ( $i = 1, \dots, g$ )

$$\alpha_i = \frac{1}{g+1} \left( \frac{1 + \zeta_{2g+2}^{2i-1}}{1 - \zeta_{2g+2}^{2i-1}} + \frac{1 + \zeta_{2g+2}}{1 - \zeta_{2g+2}} \right)$$

and for  $1 \leq k \leq i \leq g$ :

$$\begin{aligned} \pi_{k,i} &= \\ &= \frac{1}{g+1} \sum_{j=1}^k \left( \frac{1 + \zeta_{2g+2}^{2j-1}}{1 - \zeta_{2g+2}^{2j-1}} + \frac{1 + \zeta_{2g+2}^{2(i-j)+1}}{1 - \zeta_{2g+2}^{2(i-j)+1}} \right) \\ &= \frac{1}{g+1} \left( \sum_{j=1}^k \frac{1 + \zeta_{2g+2}^{2j-1}}{1 - \zeta_{2g+2}^{2j-1}} + \sum_{j=i-k+1}^i \frac{1 + \zeta_{2g+2}^{2j-1}}{1 - \zeta_{2g+2}^{2j-1}} \right). \end{aligned}$$

Theorem 2 follows from

$$\frac{1 + \zeta_{2g+2}^{2j-1}}{1 - \zeta_{2g+2}^{2j-1}} = \frac{1 + \cos \frac{2j-1}{g+1} \pi}{\sin \frac{2j-1}{g+1} \pi} i.$$

◊

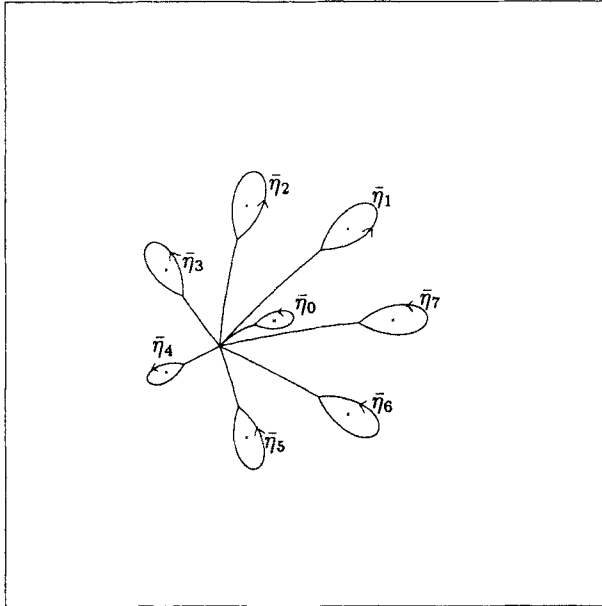


Figure 2: Curves of type  $Z_7(0,1)$  ( $g = 3$ )

### 3 Curves of type $Z_{2g+1}(0,1)$

**Theorem 3** *The period matrix of the hyperelliptic curve with equation  $w^2 = z(2g+1 - z^{2g+1})$  can be given by*

$$\pi_{i,j} = 1 - \frac{1}{\tau_1} \sum_{k=1}^i \tau_k \tau_{j-i+k} \quad (1 \leq i \leq j \leq g)$$

where

$$\tau_1 = \begin{cases} \zeta_{2g+1}^{-\frac{g}{2}} & \text{if } g \text{ even} \\ -\zeta_{2g+1}^{\frac{g+1}{2}} & \text{if } g \text{ odd} \end{cases}$$

and the reduction formula

$$\tau_{i+1} = \frac{\tau_1 (1 - \sum_{k=2}^i \zeta_{2g+1}^{g-i+k-1} \tau_k \tau_{i-k+2})}{1 + \zeta_{2g+1}^i} \quad i = 1, \dots, g-1.$$

Proof. The reduced automorphism group is generated by

$$T(z, w) = (\zeta_{2g+1} z, \zeta_{2g+1}^{g+1} w).$$

We define the symplectic basis of  $C$  in the same way as in section 2 using figure 2.

We get

$$T_* : \begin{cases} \eta_j & \mapsto \eta_{j+1} \\ \eta_{2g+1} & \mapsto \eta_1 \\ \eta_0 & \mapsto \eta_0 \end{cases} \quad (j = 1, \dots, 2g)$$

where  $\eta_1^{\eta_0^{-1}}$  is conjugation of  $\eta_1$  by the inverse of  $\eta_0$ . Thus

$$\begin{aligned} T_*\gamma_i &= [\eta_{2i}\eta_{2i+1}] = \begin{cases} \delta_i - \delta_{i+1} & i < g \\ \delta_g & i = g \end{cases} \\ T_*\delta_i &= [\eta_{2i}\eta_{2i-1} \cdots \eta_3\eta_2\eta_0] = [\eta_{2i}\eta_{2i-1} \cdots \eta_3\eta_2\eta_1\eta_1\eta_0] = \\ &= -[\eta_{2i-1}\eta_{2i}] - \dots - [\eta_3\eta_4] - [\eta_1\eta_2] + [\eta_1\eta_0] = \\ &= -\gamma_i - \dots - \gamma_2 - \gamma_1 + \delta_1. \end{aligned}$$

or

$$M = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \ddots & \ddots & -1 \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 1 \\ -1 & 0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & & \ddots & 0 & \vdots & \vdots & & \vdots & \vdots \\ -1 & \dots & \dots & \dots & -1 & 1 & 0 & \dots & \dots & 0 \end{pmatrix} \in \text{Sp}_g(\mathbf{Z})$$

and

$$L' = \begin{pmatrix} \zeta_{2g+1}^{g+1} & 0 & \dots & \dots & \dots & 0 \\ 0 & \zeta_{2g+1}^{g+2} & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & \zeta_{2g+1}^{2g-1} & 0 \\ 0 & \dots & \dots & \dots & 0 & \zeta_{2g+1}^{2g} \end{pmatrix}.$$

Consider the equation  $M^2(\Pi) = (A_2\Pi + B_2)(C_2\Pi + D_2)^{-1} = \Pi$ :

$$\begin{aligned} (-1 + \pi_{1,1})\pi_{i,j} &= (-1 + \pi_{i+1,1})(1 - \pi_{1,j+1}) \\ &\quad + \pi_{i+1,j+1}(-1 + \pi_{1,1}) \quad (1 \leq i, j \leq g-1) \quad (1) \\ (-1 + \pi_{1,1})\pi_{g,j} &= -1 + \pi_{1,j+1} \quad (1 \leq j \leq g-1) \quad (2) \\ (-1 + \pi_{1,1})\pi_{i,g} &= -1 + \pi_{i+1,1} \quad (1 \leq i \leq g-1) \quad (3) \\ (-1 + \pi_{1,1})\pi_{g,g} &= -1 \quad (4) \end{aligned}$$

Set  $\tau_i := 1 - \pi_{1,i}$  ( $i = 1, \dots, g$ ); for  $1 \leq i \leq j \leq g$  we have:

$$\tau_1(1 - \pi_{i,j}) = \sum_{k=1}^i \tau_k \tau_{j-i+k}.$$

And the  $\tau_i$ 's satisfy:

$$\begin{aligned} \tau_1 - \tau_{i+1} &= \sum_{k=1}^i \tau_k \tau_{g-i+k} \quad (i = 1, \dots, g-1) \\ \tau_1 - 1 &= \sum_{k=1}^g \tau_k^2 \end{aligned}$$

The characteristic polynomial of  $C_2\Pi + D_2$  is easily seen to be

$$\text{charpol}(C_2\Pi + D_2) = X^g + \sum_{j=0}^{g-1} \tau_{j+1} X^j.$$

Because  $L'^2$  is similar to  $C_2\Pi + D_2$  we get:

$$\prod_{j=1}^g (X - \zeta_{2g+1}^{2j-1}) = X^g + \sum_{j=0}^{g-1} \tau_{j+1} X^j.$$

There are two ways to express the polynomial  $\prod_{j=1}^g (X - \zeta_{2g+1}^{-(2j-1)})$  by means of this characteristic polynomial (here  $\zeta := \zeta_{2g+1}$ ):

$$\begin{aligned} \prod_{j=1}^g (X - \zeta^{-(2j-1)}) &= \\ &= \prod_{j=1}^g (X - \zeta \zeta^{2j-1}) \\ &= X^g + \sum_{j=0}^{g-1} \zeta^{g-j} \tau_{j+1} X^j. \end{aligned}$$

and

$$\begin{aligned} \prod_{j=1}^g (X - \zeta^{-(2j-1)}) &= \\ &= (-1)^g X^g \left( \prod_{j=1}^g \zeta^{2j-1} \right)^{-1} \prod_{j=1}^g (X^{-1} - \zeta^{2j-1}) \\ &= X^g \tau_1^{-1} (X^{-g} + \sum_{j=0}^{g-1} \tau_{j+1} X^{-j}) \\ &= \tau_1^{-1} (1 + \sum_{j=0}^{g-1} \tau_{j+1} X^{g-j}). \end{aligned}$$

Comparing the coefficients:

$$\begin{aligned} \zeta^{g-j} \tau_{j+1} &= \tau_1^{-1} \tau_{g-j+1} & (j = 1, \dots, g-1) & \quad (1) \\ \zeta^g \tau_1 &= \tau_1^{-1}. & & \quad (2) \end{aligned}$$

Thus

$$\begin{aligned} \tau_1 &= (-1)^g \prod_{j=1}^g \zeta^{2j-1} = (-1)^g \zeta^{\sum_{j=1}^g (2j-1)} \\ &= (-1)^g \zeta^{g^2} = \begin{cases} \zeta^{-\frac{g}{2}} & g \text{ even} \\ -\zeta^{\frac{g+1}{2}} & g \text{ odd} \end{cases} \\ \tau_g &= -\sum_{j=1}^g \zeta^{2j-1} = -\zeta^{\frac{\zeta^{2g}-1}{\zeta^2-1}} \\ &= \frac{\zeta^{-1}}{\zeta^2-1} = \frac{1}{\zeta+1}. \end{aligned}$$

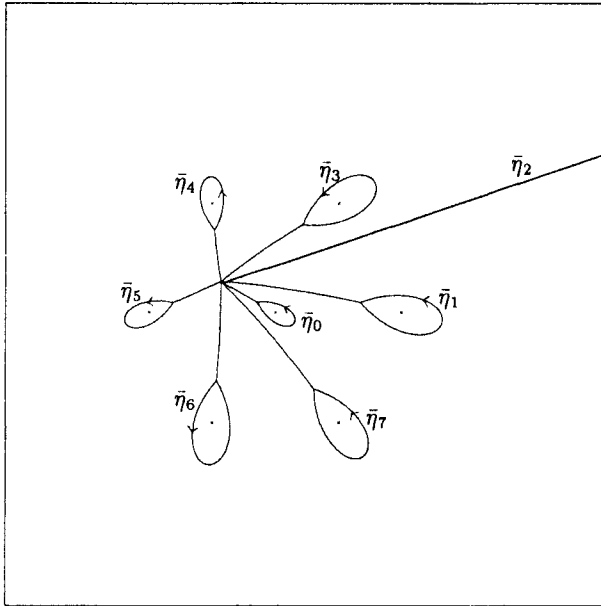


Figure 3: Curves of type  $D_6(0,1,1)$  ( $g = 3$ )

And we get the following reduction formula

$$\begin{aligned} \tau_{i+1} &= \tau_1 - \sum_{k=1}^i \tau_k \tau_{g-i+k} \\ \tau_{i+1} &= \frac{\tau_1 (1 - \sum_{k=2}^i \zeta^{g-i+k-1} \tau_k \tau_{i-k+2})}{1 + \zeta^{-i}} \end{aligned} \quad (i = 1, \dots, g-1).$$

This proves theorem 3. ◊

### 4 Curves of type $D_{2g}(0,1,1)$

**Theorem 4** *The period matrix of the hyperelliptic curve with equation  $w^2 = z(z^{2g} - 1)$  can be given by  $\Pi = (\pi_{k,j})_{k,j=1,\dots,g}$ :*

$$\begin{aligned} \pi_{1,j} &= \alpha_j \quad (j = 1, \dots, g) \\ \pi_{k,k} &= -2\alpha_2 \quad (k = 2, \dots, g) \\ \pi_{k,j} &= \alpha_{j-k+1} - \alpha_{j-k+2} \quad (2 \leq k < j \leq g) \end{aligned}$$

if  $\alpha_k \in \mathbb{C}$  is defined by

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(-\alpha_2 - \sum_{j=2}^g \alpha_j - 1) \\ \alpha_k &= \frac{2\zeta_{4g}^{2k-3}}{g(\zeta_{2g}^{2k-3} - 1)} \quad (k = 2, \dots, g) \end{aligned}$$

Proof. The reduced automorphism group is generated by  $T(z, w) = (\zeta_{2g}z, \pm\zeta_{4g}w)$  and  $T_1(z, w) = (\frac{1}{z}, \pm z^{-(g+1)}iw)$ . The symplectic basis on  $C$  is defined as in section 2 using figure 3. Here

$$T_* : \begin{cases} \eta_0 & \mapsto \eta_0 \\ \eta_1 & \mapsto \eta_3 \\ \eta_2 & \mapsto \eta_2^{\eta_3^{-1}} \\ \eta_j & \mapsto \eta_{j+1} \quad (j = 3, \dots, 2g) \\ \eta_{2g+1} & \mapsto \eta_1^{\eta_0^{-1}} \end{cases}$$

thus

$$\begin{aligned} T_*\gamma_1 &= [\eta_3\eta_3\eta_2\eta_3] = [\eta_2\eta_3] = -\delta_2 + \delta_1 \\ T_*\gamma_i &= [\eta_{2i}\eta_{2i+1}] = \begin{cases} \delta_i - \delta_{i+1} & i = 2, \dots, g-1 \\ \delta_g & i = g \end{cases} \\ T_*\delta_1 &= [\eta_3\eta_0] = [\eta_3\eta_2\eta_2\eta_1\eta_1\eta_0] = -\gamma_1 + \delta_2 \\ T_*\delta_i &= [\eta_{2i}\eta_{2i-1} \cdots \eta_4\eta_3\eta_3\eta_0] \\ &= -[\eta_{2i-1}\eta_{2i}] - \dots - [\eta_3\eta_4] + [\eta_2\eta_3] + [\eta_3\eta_0] \\ &= -\gamma_1 - \gamma_2 - \dots - \gamma_i + \delta_1 \quad (i = 2, \dots, g). \end{aligned}$$

and thus

$$M = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \ddots & \ddots & -1 \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & 1 \\ -1 & 0 & \dots & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots & 1 & 0 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 & \vdots & \vdots & & \ddots & 0 \\ -1 & \dots & \dots & \dots & -1 & 1 & 0 & \dots & \dots & 0 \end{pmatrix} \in Sp_g(\mathbf{Z})$$

Conjugation by

$$S = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & T \end{pmatrix} \in Sp_g(\mathbf{Z})$$

where

$$T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

yields

$$\tilde{M} = M^S = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 & 2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots & \vdots & 1 & \ddots & & \vdots \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & & \ddots & 0 \\ 1 & 0 & \dots & \dots & 0 & 2 & 1 & \dots & \dots & 1 \\ -1 & 1 & 0 & \dots & 0 & -1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & 0 & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 & \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & -1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

We see that

$$L' = \begin{pmatrix} \zeta_{4g} & 0 & \dots & \dots & \dots & 0 \\ 0 & \zeta_{4g}^3 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & \zeta_{4g}^{2g-3} & 0 \\ 0 & \dots & \dots & \dots & 0 & \zeta_{4g}^{2g-1} \end{pmatrix}.$$

$\tilde{M}(\Pi) = \Pi$  yields

$$\begin{aligned} \pi_{1,1} &= \pi_{1,1} + \pi_{2,1} + \pi_{1,2} + \pi_{2,2} \\ \pi_{1,j} &= \pi_{1,j+1} + \pi_{2,j+1} \quad (2 \leq j \leq g-1) \\ \pi_{1,g} &= -2\pi_{1,1} - 2\pi_{2,1} - \sum_{k=2}^g (\pi_{1,k} + \pi_{2,k}) - 1 \\ \pi_{i,1} &= \pi_{i+1,1} + \pi_{i+1,2} \quad (2 \leq i \leq g-1) \\ \pi_{i,j} &= \pi_{i+1,j+1} \quad (2 \leq i, j \leq g-1) \\ \pi_{i,g} &= -2\pi_{i+1,1} - \sum_{k=2}^g \pi_{i+1,k} \quad (2 \leq i \leq g-1) \\ \pi_{g,1} &= -2\pi_{1,1} - 2\pi_{1,2} - \sum_{k=2}^g (\pi_{k,1} + \pi_{k,2}) - 1 \\ \pi_{g,j} &= -2\pi_{1,j+1} - \sum_{k=2}^g \pi_{k,j+1} \quad (2 \leq j \leq g-1) \\ \pi_{g,g} &= 4\pi_{1,1} + 2 \sum_{k=2}^g \pi_{k,1} + \sum_{k=2}^g (2\pi_{1,k} + \sum_{m=2}^g \pi_{m,k}) + 2 \end{aligned}$$

Let again  $\alpha_i := \pi_{1,i}$ . Then we get using the above equations

$$\begin{aligned} \pi_{i,i} &= -2\alpha_2 \quad (2 \leq i \leq g) \\ \pi_{i,j} &= \alpha_{j-i+1} - \alpha_{j-i+2} \quad (2 \leq i, j \leq g). \end{aligned}$$

The  $\alpha_i$ 's satisfy the following equations

$$\alpha_{i+1} = \alpha_{g-i+2} \quad i = 2, \dots, g-1$$

$$\sum_{j=1}^g \alpha_j = -\alpha_1 - \alpha_2 - 1$$

In addition to that we see

$$\sum_{k=2}^j \pi_{i,k} = \begin{cases} \alpha_{i-j+1} - \alpha_{i+2} & j < i \\ -\alpha_{i+2} - \alpha_{j-i+2} & j \geq i \end{cases}.$$

For  $1 \leq i \leq \frac{g}{2}$

$$\text{Tr}(C_{2i-1}\Pi + D_{2i-1}) = -g\alpha_{i+1}$$

and

$$\text{Tr}(L^{2i-1}) = -\frac{2\zeta_{4g}^{2i-1}}{\zeta_{2g}^{2i-1} - 1}$$

thus

$$\alpha_i = \frac{2\zeta_{4g}^{2i-3}}{g(\zeta_{2g}^{2i-3} - 1)} \quad (i = 2, \dots, g).$$

This proves theorem 4. ◊

## References

- [1] *O. Bolza*. On Binary Sextics with Linear Transformations onto themselves, *Am. J. Math.* **10** (1888)
- [2] *W. Ingrisch*. Automorphismengruppen und Moduln hyperelliptischer Kurven, Thesis, Erlangen 1985
- [3] *H. Farkas, I. Kra*. Riemann surfaces, New York 1980
- [4] *F. Klein*. Vorlesungen über das Ikosaeder, Leipzig 1884
- [5] *H. Lange, Ch. Birkenhake*. Complex Abelian Varieties, Erlangen 1992
- [6] *W. Magnus, A. Karrass, D. Solitar*. Combinatorial Group Theory, New York 1965
- [7] *B. Schindler*. Jacobische Varietäten hyperelliptischer Kurven und einiger spezieller Kurven vom Geschlecht 3, Thesis, Erlangen 1991

Bernhard Schindler  
 Mathematisches Institut  
 Bismarkstraße 1½  
 W-8520 Erlangen  
 Germany