A primer on computational group homology and cohomology using `GAP` and SAGE*

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1-14-2008

These are expanded lecture notes of a series of expository talks surveying basic aspects of group cohomology and homology. They were written for someone who has had a first course in graduate algebra but no background in cohomology. You should know the definition of a (left) module over a (non-commutative) ring, what $\mathbb{Z}[G]$ is (where G is a group written multiplicatively and $\mathbb{Z}$ denotes the integers), and some ring theory and group theory. However, an attempt has been made to (a) keep the presentation as simple as possible, (b) either provide an explicit reference or proof of everything.

Several computer algebra packages are used to illustrate the computations, though for various reasons we have focused on the free, open source packages, such as `GAP` [Gap] and SAGE [St] (which includes `GAP`). In particular, Graham Ellis generously allowed extensive use of his HAP [E1] documentation (which is sometimes copied almost verbatim) in the presentation below. Some interesting work not included in this (incomplete) survey is (for example) that of Marcus Bishop [Bi], Jon Carlson [C] (in MAGMA), David Green [Gr] (in C), Pierre Guillot [Gu] (in GAP, C++, and SAGE), and Marc Röder [Ro].

Though Graham Ellis' `HAP` package (and Marc Röder's add-on `HAPcryst` [Ro]) can compute comhomology and homology of some infinite groups, the computational examples given below are for finite groups only.

1 Introduction

First, some words of motivation.

Let G be a group and A a G-module[1].

Let A^G denote the largest submodule of A on which G acts trivially. Let us begin by asking ourselves the following natural question.

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[*] Dedecated to my friend and colleague Tony Gaglione on the occasion of his 60^{th} birthday.

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[1] We call an abelian group A (written additively) which is a left $\mathbb{Z}[G]$-module a G-**module**.

Question: Suppose A is a submodule of a G-module B and x is an arbitrary G-fixed element of B/A. Is there an element b in B, also fixed by G, which maps onto x under the quotient map?

The answer to this question can be formulated in terms of group cohomology. ("Yes", if $H^1(G, A) = 0$.) The details, given below, will help motivate the introduction of group cohomology.

Let A_G is the largest quotient module of A on which G acts trivially. Next, we ask ourselves the following analogous question.

Question: Suppose A is a submodule of a G-module B and b is an arbitrary element of B_G which maps to 0 under the natural map $B_G \to (B/A)_G$. Is there an element a in a_G which maps onto b under the inclusion map?

The answer to this question can be formulated in terms of group homology. ("Yes", if $H_1(G, A) = 0$.) The details, given below, will help motivate the introduction of group homology.

Group cohomology arises as the right higher derived functor for $A \longmapsto A^G$. The **cohomology groups of G with coefficients in A** are defined by

$$H^n(G, A) = \mathrm{Ext}\,^n_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

(See §4 below for more details.) These groups were first introduced in 1943 by S. Eilenberg and S. MacLane [EM]. The functor $A \longmapsto A^G$ on the category of left G-modules is additive and left exact. This implies that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of G-modules then we have a **long exact sequence of cohomology**

$$\begin{array}{c} 0 \to A^G \to B^G \to C^G \to H^1(G, A) \to \\ H^1(G, B) \to H^1(G, C) \to H^2(G, A) \to \dots \end{array} \tag{1}$$

Similarly, group homology arises as the left higher derived functor for $A \longmapsto A_G$. The **homology groups of G with coefficients in A** are defined by

$$H_n(G, A) = \mathrm{Tor}\,^{\mathbb{Z}[G]}_n(\mathbb{Z}, A).$$

(See §5 below for more details.) The functor $A \longmapsto A_G$ on the category of left G-modules is additive and right exact. This implies that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of G-modules then we have a **long exact sequence of homology**

$$\begin{array}{c} \dots \to H_2(G, C) \to H_1(G, A) \to H_1(G, B) \to \\ H_1(G, C) \to A_G \to B_G \to C_G \to 0. \end{array} \tag{2}$$

Here we will define both cohomology $H^n(G, A)$ and homology $H_n(G, A)$ using projective resolutions and the higher derived functors $\mathrm{Ext}\,^n$ and $\mathrm{Tor}\,_n$. We "compute" these when G is a finite cyclic group. We also give various functorial properties, such as corestriction, inflation, restriction, and transfer. Since some of these cohomology groups can be computed with the help of computer algebra systems, we also include some discussion of how to use computers to compute them. We include several applications to group theory.

One can also define $H^1(G,A)$, $H^2(G,A)$, $\ldots$, by explicitly constructing cocycles and coboundaries. Similarly, one can also define $H_1(G,A)$, $H_2(G,A)$, $\ldots$, by explicitly constructing cycles and boundaries. For the proof that these constructions yield the same groups, see Rotman [R], chapter 10.

For the general outline, we follow §7 in chapter 10 of [R] on homology. For some details, we follow Brown [B], Serre [S] or Weiss [W].

For a recent expository account of this topic, see for example Adem [A]. Another good reference is Brown [B].

2 Differential groups

In this section cohomology and homology are viewed in the same framework. This "differential groups" idea was introduced by Cartan and Eilenberg [CE], chapter IV, and developed in R. Godement [G], chapitre 1, §2. However, we shall follow Weiss [W], chapter 1.

2.1 Definitions

A **differential group** is a pair (L,d), L an abelian group and $d: L \to L$ a homomorphism such that $d^2 = 0$. We call d a **differential operator**. The group

$$H(L) = \mathrm{Kernel}\,(d)/\mathrm{Image}\,(d)$$

is the **derived group** of (L,d). If

$$L = \oplus_{n=-\infty}^{\infty} L_n$$

then we call L **graded**. Suppose d (more precisely, $d|_{L_n}$) satisfies, in addition, for some fixed $r \neq 0$,

$$d : L_n \to L_{n+r}, \qquad n \in \mathbb{Z}.$$

We say d is **compatible** with the grading provided $r = \pm 1$. In this case, we call (L,d,r) a **graded differential group**. As we shall see, the case $r = 1$ corresponds to cohomology and the the case $r = -1$ corresponds to homology. Indeed, if $r = 1$ then we call (L,d,r) a (differential) **group of cohomology type** and if $r = -1$ then we call (L,d,r) a **group of homology type**. Note that if $L = \oplus_{n=-\infty}^{\infty} L_n$ is a group of cohomology type then $L' = \oplus_{n=-\infty}^{\infty} L'_n$ is a group of homology type, where $L'_n = L_{-n}$, for all $n \in \mathbb{Z}$.

For the impatient: For *cohomology*, we shall eventually take $L = \oplus_n \mathrm{Hom}_G(X_n, A)$, where the X_n form a chain complex (with $+1$ grading) determined by a certain type of resolution. The group $H(L)$ is an abbreviation for $\oplus_n \mathrm{Ext}^{\,n}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$. For *homology*, we shall eventually take $L = \oplus_n \mathbb{Z} \otimes_{\mathbb{Z}[G]} X_n$, where the X_n form a chain complex (with -1 grading) determined by a certain type of resolution. The group $H(L)$ is an abbreviation for $\oplus_n \mathrm{Tor}^{\,\mathbb{Z}[G]}_n(\mathbb{Z}, A)$.

Let $(L, d) = (L, d_L)$ and $(M, d) = (M, d_M)$ be differential groups (to be more precise, we should use different symbols for the differential operators of L and M but, for notational simplicity, we use the same symbol and hope the context removes any ambiguity). A homomorphism $f : L \to M$ satisfying $d \circ f = f \circ d$ will be called **admissible**. For any $n \in \mathbb{Z}$, we define $nf : L \to M$ by $(nf)(x) = n \cdot f(x) = f(x) + \cdots + f(x)$ (n times). If f is admissible then so is nf, for any $n \in \mathbb{Z}$. An admissible map f gives rise to a map of derived groups: define the map $f_* : H(L) \to H(M)$, by $f_*(x + dL) = f(x) + dM$, for all $x \in L$.

2.2 Properties

Let f be an admissible map as above.

1. The map $f_* : H(L) \to H(M)$ is a homomorphism.
2. If $f : L \to M$ and $g : L \to M$ are admissible, then so is $f + g$ and we have $(f + g)_* = f_* + g_*$.
3. If $f : L \to M$ and $g : M \to N$ are admissible then so is $g \circ f : L \to N$ and we have $(g \circ f)_* = g_* \circ f_*$.
4. If

   $$0 \to L \xrightarrow{i} M \xrightarrow{j} N \to 0 \tag{3}$$

   is an exact sequence of differential groups with admissible maps i, j then there is a homomorphism $d_* : H(N) \to H(L)$ for which the following triangle is exact:

$$\begin{tikzcd}
 & H(L) \arrow[dd, "i_*"] \\
H(N) \arrow[ur, "d_*"] & \\
 & H(M) \arrow[ul, "j_*"]
\end{tikzcd} \tag{4}$$

This diagram[2] encodes both the long exact sequence of cohomology (1) and the long exact sequence of homology (2).

Here is the construction of d_*:

Recall $H(N) = \text{Kernel}\,(d)/\text{Image}\,(d)$, so any $x \in H(N)$ is represented by an $n \in N$ with $dn = 0$. Since j is surjective, there is an $m \in M$ such that $j(m) = n$. Since j is admissible and the sequence is exact, $j(dm) = d(j(m)) = dn = 0$, so $dm \in \text{Kernel}\,(j) = \text{Image}\,(i)$. Therefore, there is an $\ell \in L$ such that $dm = i(\ell)$. Define $d_*(x)$ to be the class of ℓ in $H(L)$, i.e., $d_*(x) = \ell + dL$.

Here's the verification that d_* is well-defined:

We must show that if we defined instead $d_*(x) = \ell' + dL$, some $\ell' \in L$, then $\ell' - \ell \in dL$. Pull back the above $n \in N$ with $dn = 0$ to an $m \in M$ such that $j(m) = n$. As above, there is an $\ell \in L$ such that $dm = i(\ell)$. Represent $x \in H(N)$ by an $n' \in N$, so $x = n' + dN$ and $dn' = 0$. Pull back this n' to an $m' \in M$ such that $j(m') = n'$. As above, there is an $\ell' \in L$ such that $dm' = i(\ell')$. We know $n' - n \in dN$, so $n' - n = dn''$, some $n'' \in N$. Let $j(m'') = n''$, some $m'' \in M$, so $j(m' - m - dm'') = n' = n - j(dm'') = n' - n - dj(m'') = n' - n - dn'' = 0$. Since the sequence $L - M - N$ is exact, this implies there is an $\ell_0 \in L$ such that $i(\ell_0) = m' - m - dm''$. But $di(\ell_0) = i(d\ell_0) = dm' - dm = i(\ell') - i(\ell) = i(\ell' - \ell)$, so $\ell' - \ell \in dL$.

5. If $M = L \oplus N$ then $H(M) = H(L) \oplus H(N)$.

   proof: To avoid ambiguity, for the moment, let d_X denote the differential operator on X, where $X \in \{L, M, N\}$. In the notation of (3), j is projection and i is inclusion. Since both are admissible, we know that

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[2] This is a special case of Théorème 2.1.1 in [G].

$d_M|_L = d_L$ and $d_M|_N = d_N$. Note that $H(X) \subset X$, for any differential group X, so $H(M) = H(M) \cap L \oplus H(M) \cap N \subset H(L) \oplus H(N)$. It follows from this that that $d_* = 0$. From the exactness of the triangle (4), it therefore follows that this inclusion is an equality.

□

6. Let L, L', M, M', N, N' be differential groups. If

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "f"'] & M \arrow[r, "j"] \arrow[d, "g"'] & N \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & L' \arrow[r, "i'"] & M' \arrow[r, "j'"] & N' \arrow[r] & 0
\end{tikzcd} \tag{5}$$

is a commutative diagram of exact sequences with i, i', j, j', f, g, h all admissible then

$$\begin{tikzcd}
H(L) \arrow[r, "i_*"] \arrow[d, "f_*"'] & H(M) \arrow[d, "g_*"'] \\
H(L') \arrow[r, "i'_*"] & H(M')
\end{tikzcd}$$

commutes,

$$\begin{tikzcd}
H(M) \arrow[r, "j_*"] \arrow[d, "g_*"'] & H(N) \arrow[d, "h_*"'] \\
H(M') \arrow[r, "i'_*"] & H(N')
\end{tikzcd}$$

commutes, and

$$\begin{tikzcd}
H(N) \arrow[r, "d_*"] \arrow[d, "h_*"'] & H(L) \arrow[d, "f_*"'] \\
H(N') \arrow[r, "d_*"] & H(L')
\end{tikzcd}$$

commutes.

This is a case of Theorem 1.1.3 in [W] and of Théorème 2.1.1 in [G].

The proofs that the first two squares commute are similar, so we only verify one and leave the other to the reader. By assumption, (5) commutes and all the maps are admissible. Representing $x \in H(M)$ by $x = m + dM$, we have

$$\begin{aligned} h_*j_*(x) &= h_*(j(m)+dN) = hj(m)+dN' = gi'(m)+dN' \\ &= g_*(i'(m)+dM') = g_*i'_*(m+dM) = g_*i'_*(x), \end{aligned}$$

as desired.

The proof that the last square commutes is a little different than this, so we prove this too. Represent $x \in H(N)$ by $x = n + dN$ with $dn = 0$ and recall that $d_*(x) = \ell + dL$, where $dm = i(\ell)$, $\ell \in L$, where $j(m) = n$, for $m \in M$. We have

$$f_*d_*(x) = f_*(\ell + dL) = f(\ell) + dL'.$$

On the other hand,

$$d_*h_*(x) = d_*(h(n) + dN') = \ell' + dL',$$

for some $\ell' \in L'$. Since $h(n) \in N'$, by the commutativity of (5) and the definition of d_*, $\ell' \in L'$ is an element such that $i'(\ell') = gi(\ell)$. Since i' is injective, this condition on ℓ' determines it uniquely mod dL'. By the commutativity of (5), we may take $\ell' = f(\ell)$.

7. Let L, L', M, M', N, N' be differential graded groups with grading $+1$ (i.e., of "cohomology type"). Suppose that we have a commutative diagram, with all maps admissible and all rows exact as in (5). Then the following diagram is commutative and has exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n-1}(N) \arrow[r, "d_*"] \arrow[d, "h_*"'] & H_n(L) \arrow[r, "i_*"] \arrow[d, "f_*"'] & H_n(M) \arrow[r, "j_*"] \arrow[d, "g_*"'] & H_n(N) \arrow[r, "d_*"] \arrow[d, "h_*"'] & H_{n+1}(L) \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & H_{n-1}(N') \arrow[r, "d_*"] & H_n(L') \arrow[r, "i'_*"] & H_n(M') \arrow[r, "j'_*"] & H_n(N') \arrow[r, "d_*"] & H_{n+1}(L') \arrow[r] & \cdots
\end{tikzcd}$$

   This is Proposition 1.1.4 in [W]. As pointed out there, it is an immediate consequence of the properties, 1-6 above.

   Compare this with Proposition 10.69 in [R].

8. Let L, L', M, M', N, N' be differential graded groups with grading -1 (i.e., of "homology type"). Suppose that we have a commutative diagram, with all maps admissible and all rows exact, as in (5). Then the following diagram is commutative and has exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(N) \arrow[r, "d_*"] \arrow[d, "h_*"'] & H_n(L) \arrow[r, "i_*"] \arrow[d, "f_*"'] & H_n(M) \arrow[r, "j_*"] \arrow[d, "g_*"'] & H_n(N) \arrow[r, "d_*"] \arrow[d, "h_*"'] & H_{n-1}(L) \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & H_{n+1}(N') \arrow[r, "d_*"] & H_n(L') \arrow[r, "i'_*"] & H_n(M') \arrow[r, "j'_*"] & H_n(N') \arrow[r, "d_*"] & H_{n-1}(L') \arrow[r] & \cdots
\end{tikzcd}$$

   This is the analog of the previous property and is proven similarly.

   Compare this with Proposition 10.58 in [R].

9. Let (L, d) be a differential graded group with grading r. If $d_n = d|_{L_n}$ then $d_{n+r} \circ d_n = 0$ and
$$\dots \to L_{n-r} \overset{d_{n-r}}{\to} L_n \overset{d_n}{\to} L_{n+r} \overset{d_n}{\to} L_{n+2r} \to \dots \tag{6}$$
is exact.

10. If $\{L_n \mid n \in \mathbb{Z}\}$ is a sequence of abelian groups with homomorphisms d_n satisfying (6) then (L, d) is a differential group, where $L = \oplus_n L_n$ and $d = \oplus_n d_n$.

2.3 Homology and cohomology

When $r = 1$, we call L_n the **group of n-cochains**, $Z_n = L_n \cap \text{Kernel}\,(d_n)$ the group of n-**cocycles**, and $B_n = L_n \cap d_{n-1}(L_{n-1})$ the group of n-**coboundaries**. We call $H_n(L) = Z_n/B_n$ the n^{th} **cohomology group**. When $r = -1$, we call L_n the **group of n-chains**, $Z_n = L_n \cap \text{Kernel}\,(d_n)$ the group of n-**cycles**, and $B_n = L_n \cap d_{n+1}(L_{n+1})$ the group of n-**boundaries**. We call $H_n(L) = Z_n/B_n$ the n^{th} **homology group**.

3 Complexes

We introduce complexes in order to define explicit differential groups which will then be used to construct group (co)homology.

3.1 Definitions

Let R be a non-commutative ring, for example $R = \mathbb{Z}[G]$.

We shall define a "finite free, acyclic, augmented chain complex" of left R-modules.

A **complex** (or chain complex or R-complex with a negative grading) is a sequence of maps
$$\dots \to X_{n+1} \overset{\partial_{n+1}}{\to} X_n \overset{\partial_n}{\to} X_{n-1} \overset{\partial_{n-1}}{\to} X_{n-2} \to \dots \tag{7}$$
for which $\partial_n \partial_{n+1} = 0$, for all n. If each X_n is a free R-module with a finite basis over R (so is $\cong R^k$, for some k) then the complex is called **finite free**. If this sequence is exact then it is called an **acyclic complex**. The complex is **augmented** if there is a surjective R-module homomorphism $\epsilon : X_0 \to \mathbb{Z}$ and an injective R-module homomorphism $\mu : \mathbb{Z} \to X_{-1}$ such that $\partial_0 = \mu \circ \epsilon$, where (as usual) $\mathbb{Z}$ is regarded as a trivial R-module.

The **standard diagram** for such an R-complex is

$$\begin{tikzcd}
\dots \arrow[r] & X_2 \arrow[r, "\partial_2"] & X_1 \arrow[r, "\partial_1"] & X_0 \arrow[r, "\partial_0"] \arrow[d, "\epsilon"'] & X_{-1} \arrow[r, "\partial_{-1}"] & X_{-2} \arrow[r] & \dots \\
 & & & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[u, "\mu"'] & & \\
 & & & 0 & 0 \arrow[u] & &
\end{tikzcd}$$

Such an acyclic augmented complex can be broken up into the **positive part**

$$\cdots \to X_2 \overset{\partial_2}{\to} X_1 \overset{\partial_1}{\to} X_0 \overset{\epsilon}{\to} \mathbb{Z} \to 0,$$

and the **negative part**

$$0 \to \mathbb{Z} \overset{\mu}{\to} X_{-1} \overset{\partial_{-1}}{\to} X_{-2} \overset{\partial_{-2}}{\to} X_{-3} \to \dots \ .$$

Conversely, given a positive part and a negative part, they can be combined into a standard diagram by taking $\partial_0 = \mu \circ \epsilon$.

If X is any left R-module, let $X^* = \mathrm{Hom}_R(X, \mathbb{Z})$ be the dual R-module, where $\mathbb{Z}$ is regarded as a trivial R-module. Associated to any $f \in \mathrm{Hom}_R(X, Y)$ is the pull-back $f^* \in \mathrm{Hom}_R(Y^*, X^*)$. (If $y^* \in Y^*$ then define $f^*(y^*)$ to be $y^* \circ f : X \to \mathbb{Z}$.) Since "dualizing" reverses the direction of the maps, if you dualize the entire complex with a -1 grading, you will get a complex with a $+1$ grading. This is the **dual complex**.

When $R = \mathbb{Z}[G]$ then we call a finite free, acyclic, augmented chain complex of left R-modules, a G-**resolution**. The maps $\partial_i : X_i \to X_{i-1}$ are sometimes called **boundary maps**.

Remark 1 Using the command `BoundaryMap` in the `GAP` `CRIME` package of Marcus Bishop, one can easily compute the boundary maps of a cohomology object associated to a G-module. However, G must be a p-group.

Example 1 We use the package `HAP` [E1] to illustrate some of these concepts more concretely. Let G be a finite group, whose elements we have ordered in some way: $G = \{g_1, ..., g_n\}$.

Since a G-resolution X_* determines a sequence of finitely generated free $\mathbb{Z}[G]$-modules, to concretely describe X_* we must be able to concretely describe a finite free $\mathbb{Z}[G]$-module. In order to represent a word w in a free $\mathbb{Z}[G]$-module M of rank n, we use a list of integer pairs $w = [[i_1, e_1], [i_2, e_2], ..., [i_k, e_k]]$. The integers i_j lie in the range $\{-n, ..., n\}$ and correspond to the free $\mathbb{Z}[G]$-generators of M and their additive inverses. The integers e_j are positive (but not necessarily distinct) and correspond to the group element g_{e_j}.

Let's begin with a `HAP` computation.

```
                                        GAP
gap> LoadPackage("hap");
true
```

```
gap> G:=Group([(1,2,3),(1,2)]);;
gap> R:=ResolutionFiniteGroup(G, 4);;
```

This computes the first 5 terms of a G-resolution ($G = S_3$)

$$X_4 \xrightarrow{\delta_4} X_3 \xrightarrow{\delta_3} X_2 \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0 \to \mathbb{Z} \to 0.$$

The bounday maps δ_i are determined from the `boundary` component of the `GAP` record `R`. This record has (among others) the following components:

- `R!.dimension(k)` – the $\mathbb{Z}[G]$-rank of the module X_k,
- `R!.boundary(k, j)` – the image in X_{k-1} of the j-th free generator of X_k,
- `R!.elts` – the elements in G,
- `R!.group` is the group in question.

Here is an illustration:

```
GAP
gap> R!.group;
 Group([ (1,2), (1,2,3) ])
gap> R!.elts;
 [ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]
gap> R!.dimension(3);
 4
gap> R!.boundary(3,1);
 [ [ 1, 2 ], [ -1, 1 ] ]
gap> R!.boundary(3,2);
 [ [ 2, 2 ], [ -2, 4 ] ]
gap> R!.boundary(3,3);
 [ [ 3, 4 ], [ 1, 3 ], [ -3, 1 ], [ -1, 1 ] ]
gap> R!.boundary(3,4);
 [ [ 2, 5 ], [ -3, 3 ], [ 2, 4 ], [ -1, 4 ], [ 2, 1 ], [ -3, 1 ] ]
```

In other words, X_3 is rank 4 as a G-module, with generators $\{f_1, f_2, f_3, f_4\}$ say, and

$$\begin{aligned}
\delta_3(f_1) &= f_1 g_2 - f_1 g_1, \\
\delta_3(f_2) &= f_2 g_2 - f_2 g_4, \\
\delta_3(f_3) &= f_3 g_4 - f_3 g_1 + f_1 g_3 - f_1 g_1, \\
\delta_3(f_4) &= f_2(g_1 + g_3 + g_5) - f_3 g_3 + f_1 g_4 - f_3 g_1.
\end{aligned}$$

Now, let us create another resolution and compute the equivariant chain map between them. Below is the complete `GAP` session:

GAP

```
gap> G1:=Group([(1,2,3),(1,2)]);
Group([ (1,2,3), (1,2) ])
gap> G2:=Group([(1,2,3),(2,3)]);
Group([ (1,2,3), (2,3) ])
gap> phi:=GroupHomomorphismByImages(G1,G2,[(1,2,3),(1,2)],[(1,2,3),(2,3)]);
[ (1,2,3), (1,2) ] -> [ (1,2,3), (2,3) ]
gap> R1:=ResolutionFiniteGroup(G1, 4);
Resolution of length 4 in characteristic 0 for Group([ (1,2), (1,2,3) ]) .

gap> R2:=ResolutionFiniteGroup(G2, 4);
Resolution of length 4 in characteristic 0 for Group([ (2,3), (1,2,3) ]) .

gap> ZP_map:=EquivariantChainMap(R1, R2, phi);
Equivariant Chain Map between resolutions of length 4 .

gap> map := TensorWithIntegers( ZP_map);
Chain Map between complexes of length 4 .

gap> Hphi := Homology( map, 3);
[ f1, f2, f3 ] -> [ f2, f2*f3, f1*f2^2 ]
gap> AbelianInvariants(Image(Hphi));
[ 2, 3 ]
gap>
gap> GroupHomology(G1,3);
[ 6 ]
gap> GroupHomology(G2,3);
[ 6 ]
```

In other words, $H(\phi)$ is an isomorphism (as it should be, since the homology is independent of the resolution choosen).

3.2 Constructions

Let $R = \mathbb{Z}[G]$.

3.2.1 Bar resolution

This section follows §1.3 in [W].

Define a symbol [.] and call it the **empty cell**. Let $X_0 = R[.]$, so X_0 is a finite free (left) R-module whose basis has only 1 element. For $n > 0$, let $g_1, \dots, g_n \in G$ and define an **n-cell** to be the symbol $[g_1, \dots, g_n]$. Let

$$X_n = \oplus_{(g_1,\dots,g_n)\in G^n} R[g_1, \dots, g_n],$$

where the sum runs over all ordered n-tuples in G^n.

Define the differential operators d_n and the augmentation ϵ, as G-module maps, by

$$
\begin{aligned}
\epsilon(g[.]) &= 1, \qquad g \in G \\
d_1([g]) &= g[.] - [.], \\
d_2([g_1,g_2]) &= g_1[g_2] - [g_1 g_2] + [g_1], \\
&\vdots \\
d_n([g_1,\dots,g_n]) &= g_1[g_2,\dots,g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1,\dots,g_{i-1},g_i g_{i+1},g_{i+2},\dots,g_n] \\
&\quad + (-1)^n[g_1,\dots,g_{n-1}],
\end{aligned}
$$

for $n \geq 1$. Note that the condition $\epsilon(g[.]) = 1$ for all $g \in G$ is equivalent to saying $\epsilon([.]) = 1$. This is because ϵ is a G-module homomorphism and $\mathbb{Z}$ is a trivial G-module, so $\epsilon(g[.]) = g\epsilon([.]) = g \cdot 1 = 1$, where the (trivial) G-action on $\mathbb{Z}$ is denoted by a $\cdot$.

The X_n are finite free G-modules, with the set of all n-cells serving as a basis.

Proposition 2 *With these definitions, the sequence*

$$\cdots \to X_2 \overset{d_2}{\to} X_1 \overset{d_1}{\to} X_0 \overset{\epsilon}{\to} \mathbb{Z} \to 0,$$

is a free G-resolution.

Sometimes this resolution is called the **bar resolution**[3]. There are two other resolutions we shall consider. One is the closely related "homogeneous resolution" and the other is the "normalized bar resolution".

This simple-looking proposition is not so simple to prove. First, we shall show it is a complex, i.e., $d^2 = 0$. Then, and this is the most non-trivial part of the proof, we show that the sequence is exact.

First, we need some definitions and a lemma.

Let $f : L \to M$ and $g : L \to M$ be $+1$-graded admissible maps. We say f is **homotopic** to g if there is a homomorphism $D : L \to M$, called a **homotopy**, such that

- $D_n = D|_{L_n} : L_n \to M_{n+1}$,
- $f - g = Dd + dD$.

If $L = M$ and the identity map $1 : L \to L$ is homotopic to the zero map $0 : L \to L$ then the homotopy is called a **contracting homotopy for** L.

Lemma 3 *If L has a contracting homotopy then $H(L) = 0$.*

---

[3] This resolution is not the same as the resolution computed by `HAP` in Example 1. For details on the resolution used by `HAP`, please see Ellis [E2].

proof: Represent $x \in H(L)$ by $\ell \in L$ with $d\ell = 0$. But $\ell = 1(\ell) - 0(\ell) = dD(\ell) + Dd(\ell) = dD(\ell)$. Since $D : L \to L$, this shows $\ell \in dL$, so $x = 0$ in $H(L)$. $\square$

Next, we construct a contracting homotopy for the complex X_* in Proposition 2 with differential operator d. Actually, we shall *temporarily* let $X_{-1} = \mathbb{Z}$, $X_{-n} = 0$ and $d_{-n} = 0$ for $n > 1$, so that that the complex is infinite in both directions. We must define $D : X \to X$ such that

- $D_{-1} = D|_{\mathbb{Z}} : \mathbb{Z} \to X_0$,
- $D_n = D|_{X_n} : X_n \to X_{n+1}$,
- $\epsilon D_{-1} = 1$ on $\mathbb{Z}$,
- $d_1 D_0 + D_{-1}\epsilon = 1$ on X_0,
- $d_{n+1} D_n + D_{n-1} d_n = 1$ in X_n, for $n \geq 1$.

Define

$$\begin{aligned} D_{-n} &= 0, \qquad n > 1, \\ D_{-1}(1) &= [.], \\ D_0(g[.]) &= [g], \\ D_n(g[g_1, \dots, g_n]) &= [g, g_1, \dots, g_n], \qquad n > 0, \end{aligned}$$

and extend to a $\mathbb{Z}$-basis linearly.

Now we must verify the desired properties.

By definition, for $m \in \mathbb{Z}$, $\epsilon D_{-1}(m) = \epsilon(m[.]) = m\epsilon([.]) = m$. Therefore, ϵD_{-1} is the identity map on $\mathbb{Z}$.

Similarly,

$$\begin{aligned} &(d_1 D_0 + D_{-1}\epsilon)(g[.]) = d_1([g]) + D_{-1}(1) \\ &= g[.] - [.] + D_{-1}(1) = g[.] - [.] + [.] = g[.]. \end{aligned}$$

For the last property, we compute

$$\begin{aligned} d_{n+1} D_n(g[g_1, \dots, g_n]) &= d_{n+1}([g, g_1, \dots, g_n]) \\ &= g[g_1, \dots, g_n] - [gg_1, \dots, g_n] \\ &\quad + \sum_{i=1}^{n-1} (-1)^{i-1} [g, g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n] \\ &\quad + (-1)^{n+1} [g, g_1, \dots, g_{n-1}], \end{aligned}$$

and

$$\begin{aligned}
&D_{n-1}d_n(g[g_1,\dots,g_n])\\
&\qquad = D_{n-1}(gd_n([g_1,\dots,g_n]))\\
&\qquad = D_{n-1}(gg_1[g_2,\dots,g_n]\\
&\qquad\qquad + \sum_{i=1}^{n-1}(-1)^i g[g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n]\\
&\qquad\qquad + (-1)^n g[g_1,\dots,g_{n-1}])\\
&\qquad = [gg_1,g_2,\dots,g_n]\\
&\qquad\qquad + \sum_{i=1}^{n-1}(-1)^i [g,g_1,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n]\\
&\qquad\qquad + (-1)^n [g,g_1,\dots,g_{n-1}].
\end{aligned}$$

All the terms but one cancels, verifying that $d_{n+1}D_n + D_{n-1}d_n = 1$ in X_n, for $n \geq 1$.

Now we show $d^2 = 0$. One verifies $d_1 d_2 = 0$ directly (which is left to the reader). Multiply $d_k D_{k-1} + D_{k-2}d_{k-1} = 1$ on the right by d_k and $d_{k+1}D_k + D_{k-1}d_k = 1$ on the left by d_k:

$$d_k D_{k-1} d_k + D_{k-2} d_{k-1} d_k = d_k = d_k d_{k+1} D_k + d_k D_{k-1} d_k.$$

Cancelling like terms, the induction hypothesis $d_{k-1}d_k = 0$ implies $d_k d_{k+1} = 0$. This shows $d^2 = 0$ and hence that the sequence in Proposition 2 is exact. This completes the proof of Proposition 2. □

The above complex can be "dualized" in the sense of §3.1. This dualized complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{\mu} X_{-1} \xrightarrow{d_{-1}} X_{-2} \xrightarrow{d_{-2}} X_{-3} \to \dots \ .$$

The **standard G-resolution** is obtained by splicing these together.

3.2.2 Normalized bar resolution

Define the **normalized cells** by

$$[g_1, ..., g_n]^* = \begin{cases} [g_1, ..., g_n], & \text{if all } g_i \neq 1,\\ 0, & \text{if some } g_i = 1. \end{cases}$$

Let $X_0 = R[.]$ and

$$X_n = \oplus_{(g_1,\dots,g_n)\in G^n} R[g_1,\dots,g_n]^*, \qquad n \geq 1,$$

where the sum runs over all ordered n-tuples in G^n. Define the differential operators d_n and the augmentation map exactly as for the bar resolution.

Proposition 4 *With these definitions, the sequence*

$$\cdots \to X_2 \overset{d_2}{\to} X_1 \overset{d_1}{\to} X_0 \overset{\epsilon}{\to} \mathbb{Z} \to 0,$$

is a free G-resolution.

Sometimes this resolution is called the **normalized bar resolution**.

proof: See Theorem 10.117 in [R]. □

3.2.3 Homogeneous resolution

Let $X_0 = R$, so X_0 is a finite free (left) R-module whose basis has only 1 element. For $n > 0$, let X_n denote the $\mathbb{Z}$-module generated by all $(n+1)$-tuples $(g_0, \dots, g_n)$. Make X_i into a G-module by defining the action by $g : X_n \to X_n$ by

$$g : (g_0, ..., g_n) \longmapsto (gg_0, \dots, gg_n), \qquad g \in G.$$

Define the differential operators ∂_n and the augmentation ϵ, as G-module maps, by

$$\begin{aligned} \epsilon(g) &= 1, \\ \partial_n(g_0, \dots, g_n) &= \sum_{i=0}^{n-1} (-1)^i (g_0, \dots, g_{i-1}, \hat{g}_i, g_{i+1}, \dots, g_n), \end{aligned}$$

for $n \geq 1$.

Proposition 5 *With these definitions, the sequence*

$$\cdots \to X_2 \overset{\partial_2}{\to} X_1 \overset{\partial_1}{\to} X_0 \overset{\epsilon}{\to} \mathbb{Z} \to 0,$$

is a G-resolution.

Sometimes this resolution is called the **homogeneous resolution**.

Of the three resolutions presented here, this one is the most straightforward to deal with.

proof: See Lemma 10.114, Proposition 10.115, and Proposition 10.116 in [R]. □

4 Definition of $H^n(G, A)$

For convenience, we briefly recall the definition of Ext ^{n}. Let A be a left R-module, where $R = \mathbb{Z}[G]$, and let (X_i) be a G-resolution of $\mathbb{Z}$. We define

$$\mathrm{Ext}\,^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \mathrm{Kernel}\,(d^*_{n+1})/\mathrm{Image}\,(d^*_n),$$

where

$$d_n^* : Hom(X_{n-1}, A) \to Hom(X_n, A),$$

is defined by sending $f : X_{n-1} \to A$ to $fd_n : X_n \to A$. It is known that this is, up to isomorphism, independent of the resolution choosen. Recall $\mathrm{Ext}\,^*_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ is the right-derived functors of the right-exact functor $A \longmapsto A^G = \mathrm{Hom}_G(\mathbb{Z}, A)$ from the category of G-modules to the category of abelian groups. We define

$$H^n(G, A) = \mathrm{Ext}\,^n_{\mathbb{Z}[G]}(\mathbb{Z}, A), \tag{8}$$

When we wish to emphasize the dependence on the resolution choosen, we write $H^n(G, A, X_*)$.

For example, let X_* denote the bar resolution in §3.2.1 above. Call $C^n = C^n(G, A) = \mathrm{Hom}_G(X_n, A)$ the **group of n-cochains of G in A**, $Z^n = Z^n(G, A) = C^n \cap \mathrm{Kernel}\,(\partial)$ the group of **n-cocycles**, and $B^n = B^n(G, A) = \partial(C^{n-1})$ the group of **n-coboundaries**. We call $H^n(G, A) = Z^n/B^n$ the n^{th} **cohomology group of G in A**. This is an abelian group.

We call also define the cohomology group using some other resolution, the normalized bar resolution or the homogeneous resolution for example. If we wish to express the dependence on the resolution X_* used, we write $H^n(G, A, X_*)$. Later we shall see that, up to isomorphism, this abelian group is independent of the resolution.

The group $H_2(G, \mathbb{Z})$ (which is isomorphic to the algebraic dual group of $H^2(G, \mathbb{C}^\times)$) is sometimes called the **Schur multiplier** of G. Here $\mathbb{C}$ denotes the field of complex numbers.

We say that the group G has **cohomological dimension** n, written $cd(G) = n$, if $H^{n+1}(H, A) = 0$ for all G-modules A and all subgroups H of G, but $H^n(H, A) \neq 0$ for some such A and H.

Remark 2

- *If $cd(G) < \infty$ then G is torsion-free*[4].
- *If G is a free abelian group of finite rank then $cd(G) = rank(G)$.*
- *If $cd(G) = 1$ then G is free. This is a result of Stallings and Swan (see for example [R], page 885).*

4.1 Computations

We briefly discuss computer programs which compute cohomology and some examples of known computations.

4.1.1 Computer computations of cohomology

`GAP` [Gap] can compute some cohomology groups[5].

---

[4] This follows from the fact that if G is a cyclic group then $H^n(G, \mathbb{Z}) \neq 0$, discussed below.

[5] See §37.22 of the `GAP` manual, M. Bishop's package `CRIME` for cohomology of p-groups, G. Ellis' package `HAP` for group homology and cohomology of finite or (certain) infinite groups, and M. Röder's `HAPCryst` package (an add-on to the `HAP` package). `SAGE` [St] computes cohomology via it's `GAP` interface.

All the SAGE commands which compute group homology or cohomology require that the package `HAP` be loaded. You can do this on the command line from the main SAGE directory by typing[6]

```
sage -i gap_packages-4.4.10_3.spkg
```

Example 6 This example uses SAGE, which wraps several of the `HAP` functions.

SAGE

```
sage: G = AlternatingGroup(5)
sage: G.cohomology(1,7)
Trivial Abelian Group
sage: G.cohomology(2,7)
Trivial Abelian Group
```

This implies $H^1(A_5, GF(7)) = H^2(A_5, GF(7)) = 0$.

4.1.2 Examples

Some example computations of a more theoretical nature.

1. $H^0(G, A) = A^G$.

   This is by definition.

2. Let L/K denote a Galois extension with finite Galois group G. We have $H^1(G, L^\times) = 1$. This is often called Hilbert's Theorem 90.

   See Theorem 1.5.4 in [W] or Proposition 2 in §X.1 of [S].

3. Let G be a finite cyclic group and A a trivial torsion-free G-module. Then $H^1(G, A) = 0$.

   This is a consequence of properties given in the next section.

4. If G is a finite cyclic group of order m and A is a trivial G-module then

   $$H^2(G, A) = A/mA$$

   This is a consequence of properties given below.

   For example, $H^2(GF(q)^\times, \mathbb{C}) = 0$.

5. If $|G| = m$, $rA = 0$ and $gcd(r, m) = 1$, then $H^n(G, A) = 0$, for all $n \geq 1$.

   This is Corollary 3.1.7 in [W].

   For example, $H^1(A_5, \mathbb{Z}/7\mathbb{Z}) = 0$.

---

[6] This is the current package name - change `4.4.10_3` to whatever the latest version is on `http://www.sagemath.org/packages/optional/` at the time you read this. Also, this command assumes you are using SAGE on a machine with an internet connection.

5 Definition of $H_n(G, A)$

We say A is **projective** if the functor $B \longmapsto \mathrm{Hom}_G(A, B)$ (from the category of G-modules to the category of abelian groups) is exact. Recall, if

$$P_{\mathbb{Z}} = \cdots \to P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{\epsilon}{\to} \mathbb{Z} \to 0 \tag{9}$$

is a projective resolution of $\mathbb{Z}$ then

$$\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A) = \mathrm{Kernel}\,(d_n \otimes 1_A)/\mathrm{Image}\,(d_{n+1} \otimes 1_A).$$

It is known that this is, up to isomorphism, independent of the resolution choosen. Recall $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, A)$ are the right-derived functors of the right-exact functor $A \longmapsto A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ from the category of G-modules to the category of abelian groups. We define

$$H_n(G, A) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A), \tag{10}$$

When we wish to emphasize the dependence on the resolution, we write $H_n(G, A, P_{\mathbb{Z}})$.

Remark 3 If G is a p-group, then using the command `ProjectiveResolution` in `GAP`'s `CRIME` package, one can easily compute the minimal projective resolution of a G-module, which can be either trivial or given as a `MeatAxe`[7] module.

Since we can identify the functor $A \longmapsto A_G$ with $A \longmapsto A \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ (where $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}[G]$-module), the following is another way to formulate this definition.

If $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}[G]$-module, then a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$ is a sequence of $\mathbb{Z}[G]$-module homomorphisms

$$...\to M_n \to M_{n-1} \to ... \to M_1 \to M_0$$

satisfying:

- (Freeness) Each M_n is a free $\mathbb{Z}[G]$-module.
- (Exactness) The image of $M_{n+1} \to M_n$ equals the kernel of $M_n \to M_{n-1}$ for all $n > 0$.
- (Augmentation) The cokernel of $M_1 \to M_0$ is isomorphic to the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

The maps $M_n \to M_{n-1}$ are the boundary homomorphisms of the resolution. Setting TM_n equal to the abelian group M_n/G obtained from M_n by killing the G-action, we get an induced sequence of abelian group homomorphisms

$$...\to TM_n \to TM_{n-1} \to ... \to TM_1 \to TM_0$$

---

[7] See for example `http://www.math.rwth-aachen.de/~MTX/`.

This sequence will generally not satisfy the above exactness condition, and one defines the integral homology of G to be

$$H_n(G,\mathbb{Z}) = \mathrm{Kernel}\,(TM_n {\to} TM_{n-1})/\mathrm{Image}\,(TM_{n+1} {\to} TM_n)$$

for all $n > 0$.

5.1 Computations

We briefly discuss computer programs which compute homology and some examples of known computations.

5.1.1 Computer computations of homology

Example 7 `GAP` will compute the Schur multiplier $H_2(G,\mathbb{Z})$ using the `AbelianInvariantsMultiplier` command. To find $H_2(A_5,\mathbb{Z})$, where A_5 is the alternating group on 5 letters, type

GAP

```
gap> A5:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> AbelianInvariantsMultiplier(A5);
[ 2 ]
```

So, $H_2(A_5,\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

Here is the same computation in SAGE:

SAGE

```
sage: G = AlternatingGroup(5)
sage: G.homology(2)
Multiplicative Abelian Group isomorphic to C2
```

Example 8 The SAGE command `poincare_series` returns the Poincare series of $G \pmod p$ (p must be a prime). In other words, if you input a (finite) permutation group G, a prime p, and a positive integer n, `poincare_series(G,p,n)` returns a quotient of polynomials $f(x) = P(x)/Q(x)$ whose coefficient of x^k equals the rank of the vector space $H_k(G, ZZ/pZZ)$, for all k in the range $1 \leq k \leq n$.

SAGE

```
sage: G = SymmetricGroup(5)
sage: G.poincare_series(2,10)
 (xˆ2 + 1)/(xˆ4 - xˆ3 - x + 1)
sage: G = SymmetricGroup(3)
sage: G.poincare_series(2,10)
    1/(-x + 1)
```

This last one implies

$$\dim_{GF(2)} H_k(S_2, \mathbb{Z}/2\mathbb{Z}) = 1,$$

for $1 \leq k \leq 10$.

Example 9 Here are some more examples using SAGE's interface to `HAP`:

SAGE

```
sage: G = SymmetricGroup(5)
sage: G.homology(1)
Multiplicative Abelian Group isomorphic to C2
sage: G.homology(2)
Multiplicative Abelian Group isomorphic to C2
sage: G.homology(3)
Multiplicative Abelian Group isomorphic to C2 x C4 x C3
sage: G.homology(4)
Multiplicative Abelian Group isomorphic to C2
sage: G.homology(5)
Multiplicative Abelian Group isomorphic to C2 x C2 x C2
sage: G.homology(6)
Multiplicative Abelian Group isomorphic to C2 x C2
sage: G.homology(7)
Multiplicative Abelian Group isomorphic to C2 x C2 x C4 x C3 x C5
```

The last one means that

$$H_7(S_5, Z) = (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}).$$

SAGE

```
sage: G = AlternatingGroup(5)
sage: G.homology(1)
Trivial Abelian Group
sage: G.homology(1,7)
Trivial Abelian Group
sage: G.homology(2,7)
Trivial Abelian Group
```

This implies $H_1(A_5, \mathbb{Z}) = H_1(A_5, GF(7)) = H_2(A_5, GF(7)) = 0$.

5.1.2 Examples

Some example computations of a more theoretical nature.

1. If A is a G-module then $\mathrm{Tor}_0^{\mathbb{Z}[G]}(\mathbb{Z}, A) = H_0(G, A) = A_G \cong A/DA$.

   proof: We need some lemmas.

   Let $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ be the augmentation map. This is a ring homomorphism (but not a G-module homomorphism). Let $D = \mathrm{Kernel}\,(\epsilon)$ denote its kernel, the **augmentation ideal**. This is a G-module.

Lemma 10 *As an abelian group, D is free abelian generated by $G-1 = \{g-1 \mid g \in G\}$.*

We write this as $D = \mathbb{Z}\langle G-1\rangle$.

proof of lemma: If $d \in D$ then $d = \sum_{g\in G} m_g g$, where $m_g \in \mathbb{Z}$ and $\sum_{g\in G} m_g = 0$. Thus, $d = \sum_{g\in G} m_g(g-1)$, so $D \subset \mathbb{Z}\langle G-1\rangle$. To show D is free: If $\sum_{g\in G} m_g(g-1) = 0$ then $\sum_{g\in G} m_g g - \sum_{g\in G} m_g = 0$ in $\mathbb{Z}[G]$. But $\mathbb{Z}[G]$ is a free abelian group with basis G, so $m_g = 0$ for all $g \in G$. $\Box$

Lemma 11 *$\mathbb{Z} \otimes_{\mathbb{Z}[G]} A = A/DA$, where DA is generated by elements of the form $ga - a$, $g \in G$ and $a \in A$.*

Recall A_G denotes the largest quotient of A on which G acts trivially[8].

proof of lemma: Consider the G-module map, $A \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$, given by $a \longmapsto 1 \otimes a$. Since $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ is a trivial G-module, it must factor through A_G. The previous lemma implies $A_G \cong A/DA$. (In fact, the quotient map $q : A \to A_G$ satisfies $q(ga - a) = 0$ for all $g \in G$ and $a \in A$, so $DA \subset \text{Kernel}\,(q)$. By maximality of A_G, $DA = \text{Kernel}\,(q)$. QED) So, we have maps $A \to A_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$. By the definition of tensor products, the map $\mathbb{Z} \times A \to A_G$, $1 \times a \longmapsto 1 \cdot aDA$, corresponds to a map $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A \to A_G$ for which the composition $A_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \to A_G$ is the identity. This forces $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$. $\Box$

See also # 11 in §6.

2. If G is a finite group then $H_0(G, \mathbb{Z}) = \mathbb{Z}$.

   This is a special case of the example above (taking $A = \mathbb{Z}$, as a trivial G-module).

3. $H_1(G, \mathbb{Z}) \cong G/[G,G]$, where $[G,G]$ is the commutator subgroup of G.

   This is Proposition 10.110 in [R], §10.7.

   proof: First, we **claim**: $D/D^2 \cong G/[G,G]$, where D is as in Lemma 10. To prove this, define $\theta : G \to D/D^2$ by $g \longmapsto (g-1) + D^2$. Since $gh - 1 - (g-1) - (h-1) = (g-1)(h-1)$, it follows that $\theta(gh) = \theta(g)\theta(h)$, so θ is a homomorphism. Since D/D^2 is abelian and $G/[G,G]$ is the maximal abelian quotient of G, we must have $\text{Kernel}\,(\theta) \subset [G,G]$. Therefore, θ factors through $\theta' : G/[G,G] \to D/D^2$, $g[G,G] \longmapsto (g-1) + D^2$. Now, we construct an inverse. Define $\tau : D \to G/[G,G]$ by $g - 1 \longmapsto g[G,G]$. Since $\tau(g - 1 + h - 1) = g[G,G] \cdot h[G,G] = gh[G,G]$, it is not hard to see that this is a homomorphism. We would be essentially done (with the construction of the inverse of θ', hence the proof of the claim) if we knew $D^2 \subset \text{Kernel}\,(\tau)$. (The inverse would be the composition of the quotient $D/D^2 \to D/\text{Kernel}\,(\tau)$ with the map induced from τ, $D/\text{Kernel}\,(\tau) \to$

---

[8] Implicit in the words "largest quotient" is a universal property which we leave to the reader for formulate precisely.

$G/[G,G]$.) This follows from the fact that any $x \in D^2$ can be written as $x = (\sum_g m_g(g-1))(\sum_h m'_h(h-1)) = (\sum_{g,h} m_g m'_h (g-1)(h-1))$, so $\tau(x) = \prod_{g,h}(ghg^{-1}h^{-1})^{m_g m'_h}[G,G] = [G,G]$. QED (claim)

Next, we show $H_1(G,\mathbb{Z}) \cong D/D^2$. From the short exact sequence

$$0 \to D \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

we obtain the long exact sequence of homology

$$\begin{array}{c} \cdots \to H_1(G,D) \to H_1(G,\mathbb{Z}[G]) \to \\ H_1(G,\mathbb{Z}) \xrightarrow{\partial} H_0(G,D) \xrightarrow{f} H_0(G,\mathbb{Z}[G]) \xrightarrow{\epsilon_*} H_0(G,\mathbb{Z}) \to 0. \end{array} \tag{11}$$

Since $\mathbb{Z}[G]$ is a free $\mathbb{Z}[G]$-module, $H_1(G,\mathbb{Z}[G]) = 0$. Therefore ∂ is injective. By item # 1 above (i.e., $H_0(G,A) \cong A/DA \cong A_G$, we have $H_0(G,\mathbb{Z}) \cong \mathbb{Z}_G = \mathbb{Z}$ and $H_0(G,\mathbb{Z}[G]) \cong \mathbb{Z}[G]/D \cong \mathbb{Z}$. By (11), ϵ_* is surjective. Combining the last two statements, we find $\mathbb{Z}/\mathrm{Kernel}\,(\epsilon_*) \cong \mathbb{Z}$.This forces ϵ_* to be injective. This, and (11), together imply f must be 0. Since this forces ∂ to be an isomorphism, we are done. $\Box$

4. Let $G = F/R$ be a presentation of G, where F is a free group and R is a normal subgroup of relations. **Hopf's formula** states: $H_2(G,\mathbb{Z}) \cong (F \cap R)/[F,R]$, where $[F,R]$ is the commutator subgroup of G.

   See [R], §10.7.

   The group $H_2(G,\mathbb{Z})$ is sometimes called the **Schur multiplier** of G.

6 Basic properties of $H^n(G,A)$, $H_n(G,A)$

Let R be a (possibly non-commutative) ring and A be an R-module. We say A is **injective** if the functor $B \longmapsto \mathrm{Hom}_G(B,A)$ (from the category of G-modules to the category of abelian groups) is exact. (Recall A is projective if the functor $B \longmapsto \mathrm{Hom}_G(A,B)$ is exact.) We say A is **co-induced** if it has the form $\mathrm{Hom}_{\mathbb{Z}}(R,B)$ for some abelian group B. We say A is **relatively injective** if it is a direct factor of a co-induced R-module. We say A is **relatively projective** if

$$\begin{array}{rrr} \pi: & \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \to & A, \\ & x \otimes a \longmapsto & xa, \end{array}$$

maps a direct factor of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ isomorphically onto A. These are the G-modules A which are isomorphic to a direct factor of the induced module $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$. When G is finite, the notions of relatively injective and relatively projective coincide[9].

---

[9] These notions were introduced by Hochschild [Ho].

1. The definition of $H^n(G, A)$ does not depend on the G-resolution X_* of $\mathbb{Z}$ used.

2. If A is an projective $\mathbb{Z}[G]$-module then $H^n(G, A) = 0$, for all $n \geq 1$.

   This follows immediately from the definitions.

3. If A is an injective $\mathbb{Z}[G]$-module then $H_n(G, A) = 0$, for all $n \geq 1$.

   See also [S], §VII.2.

4. If A is a relatively injective $\mathbb{Z}[G]$-module then $H^n(G, A) = 0$, for all $n \geq 1$.

   This is Proposition 1 in [S], §VII.2.

5. If A is a relatively projective $\mathbb{Z}[G]$-module then $H^n(G, A) = 0$, for all $n \geq 1$.

   This is Proposition 2 in [S], §VII.4.

6. If $A = A' \oplus A''$ then $H^n(G, A) = H^n(G, A') \oplus H^n(G, A'')$, for all $n \geq 0$. More generally, if I is any indexing family and $A = \oplus_{i \in I} A_i$ then $H^n(G, A) = \oplus_{i \in I} H^n(G, A_i)$, for all $n \geq 0$.

   This follows from Proposition 10.81 in §10.6 of Rotman [R].

7. If

   $$0 \to A \to B \to C \to 0$$

   is an exact sequence of G-modules then we have a long exact sequence of cohomology (1). See [S], §VII.2, and properties of the *ext* functor [R], §10.6.

8. $A \longmapsto H^n(G, A)$ is the higher right derived functor associated to $A \longmapsto A^G = \mathrm{Hom}_G(A, \mathbb{Z})$ from the category of G-modules to the category of abelian groups.

   This is by definition. See [S], §VII.2, or [R], §10.7.

9. If

   $$0 \to A \to B \to C \to 0$$

   is an exact sequence of G-modules then we have a long exact sequence of homology (2). In the case of a finite group, see [S], §VIII.1. In general, see [S], §VII.4, and properties of the Tor functor in [R], §10.6.

10. $A \longmapsto H_n(G, A)$ is the higher left derived functor associated to $A \longmapsto A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ on the category of G-modules.

    This is by definition. See [S], §VII.4, or [R], §10.7.

11. If G is a finite cyclic group then

$$\begin{aligned} H_0(G,A) &= A_G, \\ H_{2n-1}(G,A) &= A^G/NA, \\ H_{2n}(G,A) &= \mathrm{Kernel}\,(N)/DA, \end{aligned}$$

for all $n \geq 1$.

To prove this, we need a lemma.

Lemma 12 *Let $G = \langle g \rangle$ be acyclic group of order k. Let $M = g-1$ and $N = 1 + g + g^2 + ... + g^{k-1}$. Then*

$$\cdots \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{M} \mathbb{Z}[G] \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{M} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

is a free G-resolution.

proof of lemma: It is clearly free. Since $MN = NM = (g-1)(1 + g + g^2 + ... + g^{k-1}) = g^k - 1 = 0$, it is a complex. It remains to prove exactness. Since Kernel $(\epsilon) = D =$ Image (M), by Lemma 10, this stage is exact.

To show Kernel (M) = Image (N), let $x = \sum_{j=0}^{k-1} m_j g^j \in$ Kernel (M). Since $(g-1)x = 0$, we must have $m_0 = m_1 = ... = m_{k-1}$. This forces $x = m_0 N \in$ Image (N). Thus Kernel $(M) \subset$ Image (N). Clearly $MN = 0$ implies Image $(N) \subset$ Kernel (M), so Kernel (M) = Image (N).

To show Kernel (N) = Image (M), let $x = \sum_{j=0}^{k-1} m_j g^j \in$ Kernel (N). Since $Nx = 0$, we have $0 = \epsilon(Nx) = \epsilon(N)\epsilon(x) = k\epsilon(x)$, so $\sum_{j=0}^{k-1} m_j = 0$. Observe that

$$\begin{aligned} x &= m_0 \cdot 1 + m_1 g + m_2 g^2 + ... + m_{k-1} g^{k-1} \\ &= (m_0 - m_0 g) + (m_0 + m_1) g + m_2 g^2 + ... + m_{k-1} g^{k-1} \\ &= (m_0 - m_0 g) + (m_0 + m_1) g - (m_0 + m_1) g^2 \\ &\quad + (m_0 + m_1 + m_2) g^2 - (m_0 + m_1 + m_2) g^3 + ... \\ &\quad + (m_0 + .. + m_{k-1}) g^{k-1} - (m_0 + .. + m_{k-1}) g^k. \end{aligned}$$

where the last two terms are actually 0. This implies $x = -M(m_0 + (m_0 + m_1)g + (m_0 + m_1 + m_2)g^2 + ... + (m_0 + .. + m_{k-1})g^{k-1} \in$ Image (M). Thus Kernel $(N) \subset$ Image (M). Clearly $NM = 0$ implies Image $(M) \subset$ Kernel (N), so Kernel (N) = Image (M).

This proves exactness at every stage.□

Now we can prove the claimed property. By property 1 in §5.1.2, it suffices to assume $n > 0$. Tensor the complex in Lemma 12 on the right with A:

$$\begin{aligned} \cdots \to \quad & \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{N_*} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{M_*} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{N_*} \\ & \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{M_*} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{\epsilon} \mathbb{Z} \otimes \mathbb{Z}[G] A \to 0, \end{aligned}$$

where the new maps are distinguished from the old maps by adding an asterisk. By definition, $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \cong A$, and by property 1 in §5.1.2, $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A \cong A/DA$. The above sequence becomes

$$\cdots \to A \overset{N_*}{\to} A \overset{M_*}{\to} A \overset{N_*}{\to} A \overset{M_*}{\to} A \overset{\epsilon}{\to} A/DA \to 0.$$

This implies, by definition of Tor,

$$\mathrm{Tor}\,^{\mathbb{Z}[G]}_{2n-1}(\mathbb{Z}, A) = \mathrm{Kernel}\,(M_*)/\mathrm{Image}\,(N_*) = A^G/NA,$$

and

$$\mathrm{Tor}\,^{\mathbb{Z}[G]}_{2n}(\mathbb{Z}, A) = \mathrm{Kernel}\,(N_*)/\mathrm{Image}\,(M_*) = A[N]/DA.$$

See also [S], §VIII.4.1 and the Corollary in §VIII.4.

12. The group $H^2(G, A)$ classifies group extensions of A by G.

    This is Theorem 5.1.2 in [W]. See also §10.2 in [R].

13. If G is a finite group of order $m = |G|$ then $mH^n(G, A) = 0$, for all $n \geq 1$.

    This is Proposition 10.119 in [R].

14. If G is a finite group and A is a finitely-generated G-module then $H^n(G, A)$ is finite, for all $n \geq 1$.

    This is Proposition 3.1.9 in [W] and Corollary 10.120 in [R].

15. The group $H^1(G, A)$ constructed using resolutions is the same as the group constructed using 1-cocycles. The group $H^2(G, A)$ constructed using resolutions is the same as the group constructed using 2-cocycles.

    This is Corollary 10.118 in [R].

16. If G is a finite cyclic group then

$$\begin{aligned} H^0(G, A) &= A^G, \\ H^{2n-1}(G, A) &= \mathrm{Kernel}\ N/DA, \\ H^{2n}(G, A) &= A^G/NA, \end{aligned}$$

for all $n \geq 1$. Here $N : A \to A$ is the norm map $Na = \sum_{g \in G} ga$ and DA is the augmentation ideal defined above (generated by elements of the form $ga - a$).

proof: The case $n = 0$: By definition, $H^0(G, A) = \mathrm{Ext}\,^0_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \mathrm{Hom}_G(\mathbb{Z}, A)$. Define $\tau : \mathrm{Hom}_G(\mathbb{Z}, A) \to A^G$ by sending $f \longmapsto f(1)$. It is easy to see that this is well-defined and, in fact, injective. For each $a \in A^G$, define $f = f_a \in \mathrm{Hom}_G(\mathbb{Z}, A)$ by $f(m) = ma$. This shows τ is surjective as well, so case $n = 0$ is proven.

Case $n > 0$: Applying the functor $\mathrm{Hom}_G(*, A)$ to the G-resolution in Lemma 12 to get

$$\cdots \leftarrow \quad \mathrm{Hom}_G(\mathbb{Z}[G], A) \overset{N_*}{\leftarrow} \mathrm{Hom}_G(\mathbb{Z}[G], A) \overset{M_*}{\leftarrow} \mathrm{Hom}_G(\mathbb{Z}[G], A) \overset{\epsilon_*}{\leftarrow} \mathrm{Hom}_G(\mathbb{Z}, A) \leftarrow 0.$$

It is known that $\mathrm{Hom}_G(\mathbb{Z}[G], A) \cong A$ (see Proposition 8.85 on page 583 of [R]). It follows that

$$\cdots \leftarrow \quad A \overset{N_*}{\leftarrow} A \overset{M_*}{\leftarrow} A \overset{\epsilon_*}{\leftarrow} A^G \leftarrow 0.$$

By definition of Ext , for $n > 0$ we have

$$\mathrm{Ext}\,^{2n}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \mathrm{Kernel}\,(M_*)/\mathrm{Image}\,(N_*) = A^G/NA,$$

and

$$\mathrm{Ext}\,^{2n-1}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \mathrm{Kernel}\,(N_*)/\mathrm{Image}\,(M_*) = \mathrm{Kernel}\,(N)/(g-1)A,$$

where g is a generator of G as in Lemma 12. □

See also [S], §VIII.4.1 and the Corollary in §VIII.4.

17. If G is a finite cyclic group of order m and A is a *trivial* G-module then

$$H^0(G, A) = A^G,$$
$$H^{2n-1}(G, A) \cong A[m],$$
$$H^{2n}(G, A) \cong A/mA,$$

for all $n \geq 1$.

This is a consequence of the previous property.

7 Functorial properties

In this section, we investigate some of the ways in which $H^n(G, A)$ depends on G.

One way to construct all these in a common framework is to introduce the notion of a "homomorphism of pairs". Let G, H be groups. Let A be a G-module and B an H-module. If $\alpha : H \to G$ is a homomorphism of groups and $\beta : A \to B$ is a homomorphism of H-modules (using α to regard B as an H-module) then we call (α, β) a **homomorphism of pairs**, written

$$(\alpha, \beta) : (G, A) \to (H, B).$$

Let $G \subset H$ be groups and A an H-module (so, by restriction, a G-module). We say a map

$$f_{G,H} : H^n(G,A) \to H^n(H,A),$$

is **transitive** if $f_{G_2,G_3} f_{G_1,G_2} = f_{G_1,G_2}$, for all subgroups $G_1 \subset G_2 \subset G_3$.

Let X_* be a G-resolution and X'_* a H-resolution, each with a -1 grading. Associated to a homomorphism of groups $\alpha : H \to G$ is a sequence of H-homomorphisms

$$A_n : X'_n \to X_n, \tag{12}$$

$n \geq 0$, such that $d_{n+1}A_{n+1} = A_n d'_{n+1}$ and $\epsilon A_0 = \epsilon'$.

Theorem 13 *1. If $(\alpha, \beta) : (G, A) \to (G', A')$ and $(\alpha', \beta') : (G', A') \to (G'', A'')$ are homomorphisms of pairs then so is $(\alpha' \circ \alpha, \beta' \circ \beta) : (G, A) \to (G'', A'')$.*

2. Suppose $(\alpha, \beta) : (G, A) \to (G', A')$ is homomorphism of pairs, X_* is a G-resolution, and X'_* is a G'-resolution (each infinite in both directions, with a -1 grading). Let $H^n(G, A, X_*)$ denote the derived groups associated to the differential groups $\mathrm{Hom}_G(X_*, A)$ with $+1$ grading. There is a homomorphism

$$(\alpha, \beta)_{X_*, X'_*} : H^n(G, A, X_*) \to H^n(G', A', X'_*)$$

satisfying the following properties.

(a) If $G = G'$, $A = A'$, $X = X'$, $\alpha = 1$ and $\beta = 1$ then $(1, 1)_{X_*, X'_*} = 1$.

(b) If $(\alpha', \beta') : (G', A') \to (G'', A'')$ is homomorphism of pairs, X''_* is a G''-resolution then

$$(\alpha' \circ \alpha, \beta' \circ \beta)_{X_*, X''_*} = (\alpha', \beta')_{X'_*, X''_*} \circ (\alpha, \beta)_{X_*, X'_*}.$$

(c) If $(\alpha, \gamma) : (G, A) \to (G', A')$ is homomorphism of pairs then

$$(\alpha, \beta + \gamma)_{X_*, X'_*} = (\alpha, \beta)_{X_*, X'_*} + (\alpha, \gamma)_{X_*, X'_*}.$$

Remark 4 *For an analogous result for homology, see §§III.8 in Brown [B].*

proof: We sketch the proof, following Weiss, [W], Theorem 2.1.8, pp 52-53.

(1): This is “obvious”.

(2): Let $(\alpha, \beta) : (G, A) \to (G', A')$ be a homomorphism of pairs. Using (12), we have an associated chain map

$$\alpha^* : \mathrm{Hom}_G(X_*, A) \to \mathrm{Hom}_{G'}(X'_*, A')$$

of differential groups (Brown §III.8 in [B]). The homomorphism of cohomology groups induced by α^* is denoted

$$\alpha^*_{n, X_*, X'_*} : H^n(G, A, X_*) \to H^n(G', A', X'_*).$$

Properties (a)-(c) follow from §2.2 and the corresponding properties of α^*. □

As the cohomology groups are independent of the resolution used, the map $(\alpha,\beta)_{X_*,X'_*} : H^n(G,A,X_*) \to H^n(G',A',X'_*)$ is sometimes simply denoted by

$$(\alpha,\beta)_* : H^n(G,A) \to H^n(G',A'). \tag{13}$$

7.1 Restriction

Let $X_* = X_*(G)$ denote the bar resolution.

If H is a subgroup of G then the cycles on G, $C^n(G,A) = \mathrm{Hom}_G(X_n(G),A)$, can be restricted to H: $C^n(H,A) = \mathrm{Hom}_H(X_n(H),A)$. The restriction map $C^n(G,A) \to C^n(H,A)$ leads to a map of cohomology classes:

$$\mathrm{Res} : H^n(G,A) \to H^n(H,A).$$

In this case, the homomorphism of pairs is given by the inclusion map $\alpha : H \to G$ and the identity map $\beta : A \to A$. The map Res is the induced map defined by (13). By the properties of this induced map, we see that $\mathrm{Res}_{H,G}$ is transitive: if $G \subset G' \subset G''$ then[10]

$$\mathrm{Res}_{G',G} \circ \mathrm{Res}_{G'',G'} = \mathrm{Res}_{G'',G}.$$

A particularly nice feature of the restriction map is the following fact.

Theorem 14 *If G is a finite group and G_p is a p-Sylow subgroup and if $H^n(G,A)_p$ is the p-primary component of $H^n(G,A)$ then*

(a) there is a canonical isomorphism $H^n(G,A) \cong \oplus_p H^n(G,A)_p$, and

(b) Res : $H^n(G,A) \to H^n(G_p,A)$ restricted to $H^n(G,A)_p$ (identified with a subgroup of $H^n(G,A)$ via (a)) is injective.

proof: See Weiss, [W], Theorem 3.1.15. □

Example 15 Homology is a functor. That is, for any $n > 0$ and group homomorphism $f : G \to G'$ there is an induced homomorphism $H_n(f) : H_n(G,\mathbb{Z}) \to H_n(G',\mathbb{Z})$ satisfying

- $H_n(gf) = H_n(g)H_n(f)$ for group homomorphisms $f : G \to G'$ $g : G' \to G''$,
- $H_n(f)$ is the identity homomorphism if f is the identity.

The following commands compute $H_3(f) : H_3(P,\mathbb{Z}) \to H_3(S_5,\mathbb{Z})$ for the inclusion $f : P \hookrightarrow S_5$ into the symmetric group S_5 of its Sylow 2-subgroup. They also show that the image of the induced homomorphism $H_3(f)$ is precisely the Sylow 2-subgroup of $H_3(S_5,\mathbb{Z})$.

---

[10] There is an analog of the restriction for homology which also satisfies this transitive property (Proposition 9.5 in Brown [B]).

GAP

```
gap>  S_5:=SymmetricGroup(5);;  P:=SylowSubgroup(S_5,2);;
gap>  f:=GroupHomomorphismByFunction(P,S_5, x->x);;
gap>  R:=ResolutionFiniteGroup(P,4);;
gap>  S:=ResolutionFiniteGroup(S_5,4);;
gap>  ZP_map:=EquivariantChainMap(R,S,f);;
gap>  map:=TensorWithIntegers(ZP_map);;
gap>  Hf:=Homology(map,3);;
gap>  AbelianInvariants(Image(Hf));
[2,4]
gap>  GroupHomology(S_5,3);
[2,4,3]
```

If H is a subgroup of finite index in G then there is an analogous restriction map in group homology (see for example Brown [B], §III.9).

7.2 Inflation

Let X_* denote the bar resolution of G. Recall

$$X_n = \oplus_{(g_1,\dots,g_n)\in G^n} R[g_1,\dots,g_n],$$

where the sum runs over all ordered n-tuples in G^n. If H is a subgroup of G, let X_*^H denote the complex defined by

$$X_n^H = \oplus_{(g_1H,\dots,g_nH)\in (G/H)^n} R[g_1H,\dots,g_nH].$$

This is a resolution, and we have a chain map defined on n-cells by $[g_1,\dots,g_n] \longmapsto [g_1H,\dots,g_nH]$.

Suppose that H is a normal subgroup of G and A is a G-module. We may view A^H as a G/H-module. In this case, the homomorphism of pairs is given by the quotient map $\alpha : G \to G/H$ and the inclusion map $\beta : A^H \to A$. The **inflation** map Inf is the induced map defined by (13), denoted

$$\mathrm{Inf}\ : H^n(G/H, A^H) \to H^n(G, A).$$

The **inflation-restriction sequence in dimension** n is

$$0 \to H^n(G/H, A^H) \overset{\mathrm{Inf}}{\to} H^n(G, A) \overset{\mathrm{Res}}{\to} H^n(H, A).$$

For a proof, see Weiss, [W], §3.4.

There an analog of this inflation-restriction sequence for homology.

We omit any discussion of transfer and Shapiro's lemma, due to space limitations.

Acknowledgements: I thank G. Ellis, M. Mazur and J. Feldvoss, P. Guillot for correspondence which improved the content of these notes.

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