

Fourier series using SAGE

1. Let

$$f(x) = \begin{cases} 2, & 3/2 \leq x \leq 3, \\ 0, & 0 \leq x \leq 3/2, \\ -1, & -3 < x < 0, \end{cases}$$

period 6.

- Find the Fourier series of $f(x)$ and graph the function the FS converges to, $-3 \leq x \leq 3$.
- Write down the series in summation notation and compute the first 3 non-zero terms.

The plot of $f(x)$ is given in Figure 1.

solution: Of course the Fourier series of $f(x)$ with period $2L$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

where a_n and b_n are

$$a_n = \frac{1}{L} \int_{-L}^L \cos(\frac{n\pi x}{L}) f(x) dx,$$

$$b_n = \frac{1}{L} \int_{-L}^L \sin(\frac{n\pi x}{L}) f(x) dx.$$

Generally, the following SAGE/Python functions compute the Fourier series coefficients of $f(x)$:

```
def fourier_series_cos_coeff(fcn,n,L): # n>= 0
    lowerlimit = -L
    upperlimit = L
    an = maxima('tldfint(%s*cos(%s*n*x/%s),x,%s,%s)/L'%(fcn,maxima(pi),L,lowerlimit,upperlimit))
    return str(an).replace("%","")

def fourier_series_sin_coeff(fcn,n,L): # n > 0
    lowerlimit = -L
    upperlimit = L
    bn = maxima('tldfint(%s*cos(%s*n*x/%s),x,-%s,%s)/L'%(fcn,maxima(pi),L,lowerlimit,upperlimit))
    return str(bn).replace("%","")
```

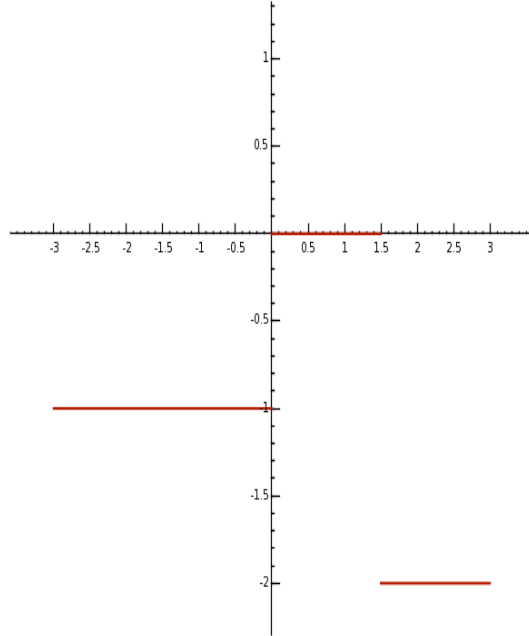


Figure 1: Piecewise polynomial function $f(x)$.

However, Maxima's support for piecewise defined functions is rudimentary and the above functions will not give us what we want. So we compute them using SAGE's "piecewise polynomial" class, with some help from Maxima:

```
sage: f1 = lambda x:2
sage: f2 = lambda x:0
sage: f3 = lambda x:-1
sage: f = Piecewise([[(-3,0),f3],[(0,3/2),f2],[(3/2,3),f1]])
sage: P = f.plot_fourier_series_partial_sum(10,3,-5,5)
sage: P.show()
sage: f.fourier_series_partial_sum(5,3)
'0 + ((-2/pi)*cos(1*pi*x/3) + (4/pi)*sin(1*pi*x/3)) + (0*cos(2*pi*x/3) + (-2/pi)*sin(2*pi*x/3))
+ ((2/(3*pi))*cos(3*pi*x/3) + (4/(3*pi))*sin(3*pi*x/3)) + (0*cos(4*pi*x/3) + 0*sin(4*pi*x/3))'
```

The plot P produced in this SAGE code is displayed in Figure 2.

The "piecewise" class requires that the subscript of the Fourier coefficient be a fixed integer. For a symbolic argument, use the following

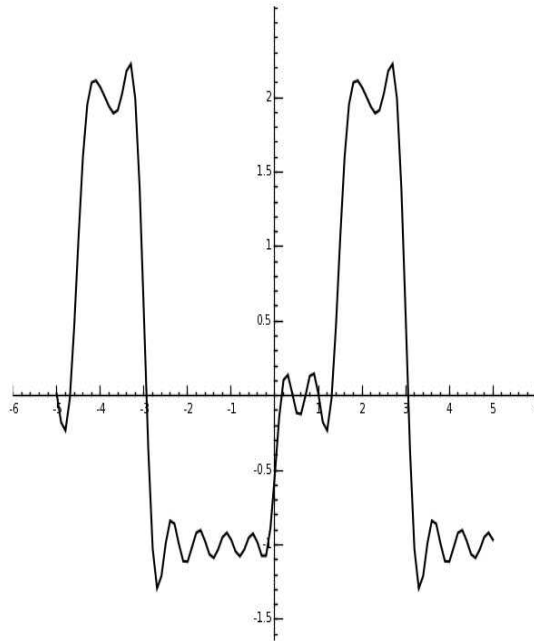


Figure 2: Graph of 1st 10 terms of partial Fourier series of $f(x)$.

commands:

```
## n > 0 FS cosine coeff
fcn = 2; lowerlimit = 3/2; upperlimit = 3; L = 3
an1 = maxima('tldefint(%s*cos(%s*n*x/%s),x,%s,%s)'%(fcn,maxima(pi),L,lowerlimit,upperlimit)); an1
# 6*sin(%pi*n)/(%pi*n) - 6*sin(%pi*n/2)/(%pi*n)
fcn = -1; lowerlimit = -3; upperlimit = 0; L = 3
an2 = maxima('tldefint(%s*cos(%s*n*x/%s),x,%s,%s)'%(fcn,maxima(pi),L,lowerlimit,upperlimit)); an2
# -3*sin(%pi*n)/(%pi*n)
an = (an1+an2)/L
# (3*sin(%pi*n)/(%pi*n) - 6*sin(%pi*n/2)/(%pi*n))/3
```

In other words, for $n > 0$, $a_n = 2 \frac{\sin(\pi n/2)}{\pi n}$. We must compute the $n = 0$ term separately:

```
## n = 0 FS cosine coeff
fcn = 2; lowerlimit = 3/2; upperlimit = 3; L = 3
an1 = maxima('tldefint(%s*cos(%s*0*x/%s),x,%s,%s)'%(fcn,maxima(pi),L,lowerlimit,upperlimit)); an1
# 3
fcn = -1; lowerlimit = -3; upperlimit = 0; L = 3
an2 = maxima('tldefint(%s*cos(%s*0*x/%s),x,%s,%s)'%(fcn,maxima(pi),L,lowerlimit,upperlimit)); an2
# -3
an = (an1+an2)/L
# 0
```

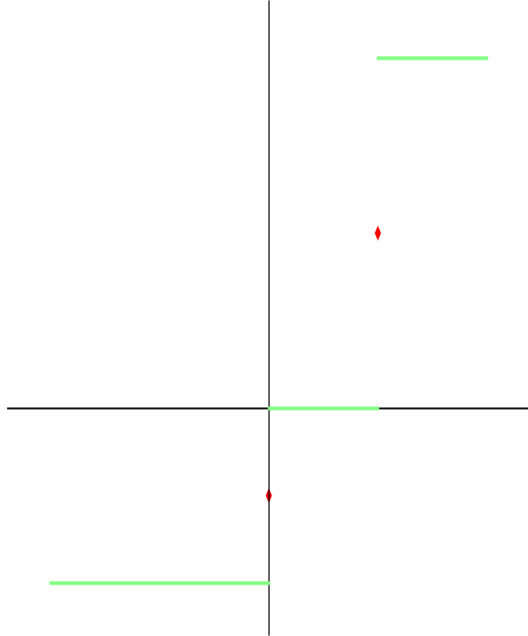



Figure 3: Graph of Fourier series of $f(x)$.

2. Let

$$f(x) = \begin{cases} 2, & 3/2 \leq t \leq 3, \\ 0, & 0 \leq t \leq 3/2, \end{cases}$$

- Find the Cosine series of $f(x)$ (period 6) and graph the function the CS converges to, $-3 \leq x \leq 3$.
- Write down the series in summation notation and compute the first 3 non-zero terms.

solution: Of course the Cosine series of $f(x)$ with period $2L$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where a_n is

$$a_n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x) dx.$$

A simple Python program:

```
def cosine_series_coeff(fcn,n,L): # n>= 0
    lowerlimit = 0
    upperlimit = L
    an = maxima('2*tldefint(%s*cos(%s*n*x/%s),x,%s,%s)/L'%(fcn,maxima(pi),L,lowerlimit,upperlimit))
    return str(an).replace("%","")
```

It was noted above that this program will not help us, for the type of function we are dealing with here. So, we have SAGE do the computations “piece-by-piece”:

```
## n > 0
fcn = 2; lowerlimit = 3/2; upperlimit = 3; L = 3
an1 = maxima('tldefint(%s*cos(%s*n*x/%s),x,%s,%s)%(fcn,maxima(pi),L,lowerlimit,upperlimit)); an1
# 6*sin(%pi*n)/(%pi*n) - 6*sin(%pi*n/2)/(%pi*n)
an = (2/L)*an1
an
# 2*(6*sin(%pi*n)/(%pi*n) - 6*sin(%pi*n/2)/(%pi*n))/3
```

In other words, $a_n = 4 \frac{\sin(\pi n/2)}{\pi n}$. To find the 0-th coefficient, use the commands

```
## n = 0
an1 = maxima('tldefint(%s*cos(%s*0*x/%s),x,%s,%s)%(fcn,maxima(pi),L,lowerlimit,upperlimit)); an1
# 3
an = (2/L)*an1
an
# 2
# a0 = 2
```

or the command

```
sage: f1 = lambda x:2
sage: f2 = lambda x:0
sage: f = Piecewise([[0,3/2],f2],[3/2,3],f1])
sage: f.cosine_series_coefficient(0,3)
2
```

In either case, $a_0 = 2$.

The Cosine series is therefore

$$f(x) \sim 1 + \sum_{n=1}^{\infty} \left(4 \frac{\sin(\pi n/2)}{\pi n}\right) \cos\left(\frac{n\pi x}{3}\right).$$

```

sage: an = lambda n:4*sin(Pi*n/2)/(Pi *n)
sage: an(1); an(2); an(3)
1.2732395447351628
0.000000000000000077963436650387510
-0.42441318157838759

```

So, $a_1 = \frac{4}{\pi} = 1.273239544735162\dots$, $a_2 = 0$, $a_3 = -\frac{4}{3\pi}$.

The function the Cosine series of $f(x)$ converges to (by Dirichlet's theorem) is in Figure 4.

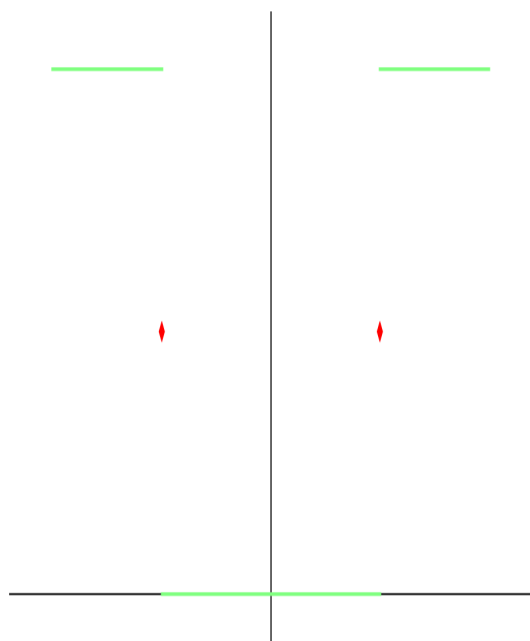


Figure 4: Graph of Cosine series of $f(x)$.

3. Let

$$f(x) = \begin{cases} 2, & 3/2 \leq t \leq 3, \\ 0, & 0 \leq t \leq 3/2, \end{cases}$$

- Find the Sine series of $f(x)$ (period 6) and graph the function the FS converges to, $-6 \leq x \leq 6$.
- Write down the series in summation notation and compute the first 3 non-zero terms.

We leave this one as an exercise for the reader!

